ON THE EXISTENCE OF SEQUENTIAL EQUILIBRIUM: THE BACKWARD INDUCTION APPROACH★

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Abstract. We generalize the well-known backward induction procedure to the case of extensive games with perfect recall having certain information structure (called simple information structure). We prove that the backward induction assessments are precisely sequential equilibria, and show that the set of backward induction assessments for any game of simple information structure is nonempty. This provides a direct proof of the existence of sequential equilibria in such games. The described procedure suggests a direct method of computing sequential equilibria for such games.

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1. Introduction

Sequential equilibria are extensively used in Economics, Finance and many other disciplines. They incorporate optimal play starting from every information set. It was conjectured by Kreps and Wilson [8] that there exists a roll-back procedure to find all sequential equilibria, however we are not aware of any successful attempt to formally describe such procedure for an arbitrary extensive form game of perfect recall. The complications arise since in general the information structure of an extensive game of imperfect information can be very complicated.

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The well-known backward induction procedure works only for the extensive games of perfect information. In such games, at any decision node the optimal choice of the decisive player does not depend on the previous choices of the players, that is, on choices made at any predecessor of that node.

The main obstacle in generalizing the standard backward induction procedure and applying it to find sequential equilibria in extensive games of imperfect information is exactly the following: in general the optimal choice at a given information set depends on the previous choices via beliefs over the nodes of that information set.

In some cases the optimal choices are independent even despite the presence of non-singleton information sets (see example below).

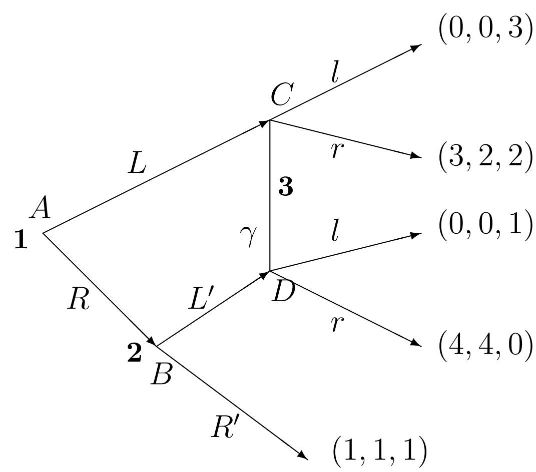

Figure 1

Example 1.1. Consider the game depicted in Figure 1. As can be seen, regardless of the belief system at the information set γ (i.e., no matter what choices players 1 and 2 made at their decision nodes), in the subform starting at γ player 3 would always choose l with probability one.

Truncate the game tree at γ, assigning the payoff vector $(0, 0, 3)$ to point C, and the payoff vector $(0, 0, 1)$ to node D. The reduced game is solved in the usual way, applying the standard backward induction procedure. This leads to player 2 selecting R' with probability 1 at his decision node B, and player 1 selecting R at node A.

In this particular case the generalization of the backward induction procedure is via using an information set in the role of a decision node within the standard backward induction method.

The main contribution of our paper is generalizing the well-known backward induction procedure to the class of extensive game with perfect recall having certain information structure (so-called games of simple information structure), which suggests a direct computation method to find all sequential equilibria.

The assumption of simple information structure is not as restrictive as it might seem. Most games used in Economics have simple information structure, including various models of sequential bargaining (see for instance [2, 10]), Entry Deterrence Game (see for example [10, 15]), various signaling games (for instance, Beer-Quiche Game in [4]), Joint Venture Entry Game [10], etc.

The paper is organized as follows. In the Section 2 we provide all necessary definitions regarding sequential games. In Section 3 we distinguish games of two different types based on their information structure: games of simple and complex information structure. In Section 3 we define two (collectively exhaustive) classes of sequential games with perfect recall: games of simple and complex information structures. In Section 5 we describe the generalized backward induction procedure for games of simple information structure. We show that this procedure yields a nonempty set of assessments, which provides (for the first time) a direct proof of the existence of sequential equilibria for sequential games of simple information structure.

2. Definitions and Methodology

As usual, a reflexive, antisymmetric and transitive binary relation on a set X is called a *partial order*.

Definition 2.1. *A **pograph** is a pair $(X, \succeq)$, where X is a finite set of nodes, and $\succeq$ is a partial order on X. The arbitrary node of X will be denoted by t.*

Intuitively, the binary relation $\succeq$ designates precedence, and the notation $t_1 \succeq t_2$ informs us that t_2 is among the successors of t_1.

Definition 2.2. *If $y \succeq x$, then y is a **predecessor** of x, and x is a **successor** of y.*

Definition 2.3. *A node $t \neq x$ is an **immediate predecessor** (or a **parent**) of x if $t \succ x$ and there is no other node s such that $t \succ s \succ x$. In this case x is called an **immediate successor**, or a **child** of t.*

A node with nonempty set of children would be referred to as a **decision node**, while a node having no children would be called a **terminal node**. A node with no parent would be called a **root**. Denote the set of all decision nodes by Y, the set of all terminal nodes by Z, and the set of roots by W. Clearly, $W \subseteq Y$ and $X = Y \cup Z$.

Definition 2.4. *If $x \succ y$, then a **path** from x to y is a chain*

$$x = x_1 \succeq x_2 \succeq \ldots \succeq x_n = y$$

such that x_{i-1} is an immediate predecessor of x_i for each $i = 1, \ldots, n$.

Definition 2.5. *A **frame** T is a pograph such that for every non-root node x there exists a unique root w such that:*

(1) *there exists a unique path from w to x, and*
(2) *for any $y \in W$ different from w, there is no path from y to x.*

Note that this definition implies that every non-root node has exactly one parent. A frame is called **finite** if the set of nodes X is finite. In this paper we consider only finite frames.

Definition 2.6. *An **n-player extensive form (or sequential) game** G is a finite game in extensive form, which is a tuple (T, P, U, C, H), whose elements are interpreted as follows.*

*A finite **frame** T is as described in Definition 2.5. We work with frames instead of trees to account for the possibility of multiple roots.*

*Denote by $\mathcal{P}$ a **player partition**, $\mathcal{P} = \{\mathcal{P}_0, \mathcal{P}_1, \ldots \mathcal{P}_n\}$. Each $\mathcal{P}_i$ is called player i's set and is a set of all nodes at which player i is decisive. Player 0 incorporates a random mechanism that may determine the game path, and may be inactive,*

*An **information partition** is denoted by U, which is a refinement of the player partition $\mathcal{P}$. It partitions each set $\mathcal{P}_i$ into information sets u. Denote the set of all information sets of player i by U_i. Each $u \in U_i$ has the property that:*

(1) *if $a, b \in u$, then $a \not\succ b$, and*
(2) *for every $a \in u$ the set of choices available at a is the same.*

Given a decision node $x \in Y$, we will denote the information set containing x by $U(x)$.

*An information partition has a property that there is an information set I_W, called the **root information set**, which consists of nodes that have no predecessors, i.e., it is the set $\{w \in W\}$. A probability measure ρ on I_W is specified.*

*Denote by C a **choice partition** that partitions the (finite) set of all choices available throughout the game, M into the subsets C_x, $x \in X$, where C_x represents the set of all choices (actions) available at decision node x. It satisfies the following condition: for every $u \in U$ and $x, y \in u$, we have $C_x = C_y$. Thus we can introduce another partition C', which partitions the set of all choices for the game into the subsets C_u, each of those containing all choices available at the information set u.*

To each choice c available at a decision node x there corresponds a unique edge originating from x, and vice versa.

*A **payoff function** H is a vector-valued function that assigns to every terminal node $z \in Z$ a vector $H(z) = (H_1(z), H_2(z), \cdots, H_n(z))$, whose components are the payoffs of players $1, \cdots, n$ at the terminal node z.*

Definition 2.7. *Given a sequential game $G = (T, P, U, C, H)$, the quadruple $\Xi = (T, P, U, C)$ is called an **extensive form** of the game G.*

We consider extensive games with perfect recall only. Let us introduce the following notation. Let D be a subset of $X \times X$ consisting of all $(x, y) \in X \times X$ such that $x \succ y$. Let $\alpha : D \twoheadrightarrow M$ be a function defined as $\alpha(x, y) = c$, where $c \in C_x$ is the choice at x that lies on the path from x to y.

Definition 2.8. *An extensive game G is called an extensive game with **perfect recall** if for every player i the following condition is satisfied: for every $u, v \in U_i$, $z, t \in u$ and $x, y \in v$, if $z \succ x$ and $t \succ y$, then $\alpha(z, x) = \alpha(t, y)$.*

Definition 2.9. *Given an information set $u \in U_i$ of player i, define a **local strategy** b_{iu} to be a probability distribution over C_u. Denote the set of all local strategies of player i at u by $B_{iu} = \Delta_{d_u}$, where d_u is a cardinality of C_u (the number of choices available at u), and Δ_{d_u} is a unit $(d(u) - 1)$-simplex.*

A local strategy b_{iu} is called *completely mixed* if $b_{iu} \in B^{\circ}_{iu}$, i.e., if every choice at C_u is played with some positive probability.

Definition 2.10. *A **behavior strategy** b_i of player i is a tuple $(b_{iu})_{u \in U_i}$, i.e., an assignment of some local strategy b_{iu} to every $u \in U_i$. The set of all behavior strategies of player i is denoted by B_i, $B = \prod_{u \in U_i} B_{iu}$.*

Definition 2.11. *A **behavior strategy combination** $b = (b_1, \cdots, b_n)$ is an n-tuple whose i^{th} component is a behavior strategy of player i.*

We will call a behavior strategy b_i *completely mixed* if for each $u \in U_i$, b_{iu} is completely mixed. A behavior strategy combination b is *completely mixed* if each b_i is completely mixed.

Fix a behavior strategy combination $b \in B$. It induces a probability measure P on Z as follows. Fix a terminal node $z \in Z$, without loss of generality let $w \in W$ be the unique root predecessor of z, and $w \succ x_1 \succ \cdots \succ x_{r-1} \succ x_r = z$ be the path from w to z. Given a non-root node x, denote by $b(x)$ the probability assigned by b to the edge connecting x with its parent (recall that in a frame every non-root node has exactly one parent). Then the realization probability of z given $b \in B$ is:

$$P(z|b) = \rho(w) \cdot \prod_{j=1}^{r} b(x_j).$$

Now we can define the expected payoff of player i given a behavior strategy combination b. Assume without loss of generality the set of terminal nodes is $Z = \{z_1, \ldots, z_m\}$. Then the expected payoff of player i can be calculated as follows:

$$h_i = \sum_{j=1}^{m} H_i(z_j) P(z_j).$$

Definition 2.12. *A* ***system of beliefs*** μ *is a function that prescribes to every non-singleton information set* $u \in U$ *a probability measure over the nodes in* u*.*

Given a sequential game G with perfect recall, the set of all beliefs systems will be denoted by M.

Definition 2.13. *An* ***assessment*** *is a pair* (μ, b)*, where* $\mu \in M$ *is a system of beliefs and* $b \in B$ *is a behavior strategy combination.*

Fix a completely mixed behavior strategy profile $b \in B^{\circ}$. Then, every node of the extensive form is reached with some positive probability. As Kreps and Wilson [8] argue, given $b \in B^{\circ}$ reasonable beliefs are those computed from b via Bayes' rule. That is, given a non-root decision node x,

$$\mu(x) = \frac{P(x|b)}{P(u(x)|b)} = \frac{P(x|b)}{\sum_{y \in u(x)} P(y|b)}. \tag{2.1}$$

Denote by $\Psi^{\circ} \subseteq B^{\circ} \times M$ the set of all assessments $\mu, b)$ such that b is a completely mixed behavior strategy profile and μ is computed from b via the above formula. Let Ψ be the closure of Ψ° in $B \times M$.

Definition 2.14. *An assessment* (μ, b) *is* ***consistent*** *if* $(\mu, b) \in \Psi$ *(i.e.,* (μ, b) *is a limit point of some sequence* $(\mu_k, b_k) \subseteq \Psi^0$*).*

Definition 2.15. *A* ***belief correspondence*** $\phi : B \twoheadrightarrow M$ *is defined for each* $b \in B$ *as* $\phi(b) = \{\mu \in M : (\mu, b) \in \Psi\}$*.*

Clearly, ϕ is nonempty-valued and closed (i.e., has a closed graph). Note that it is a function on B° (because for any completely mixed behavior strategy profile, every information set is reached with some positive probability, so that the ratio in Equation 2.1 is well-defined for each non-root decision node).

For each information set $u \in U$, denote by $Z(u) \subseteq Z$ the set of all terminal successors of u. A node $z \in Z$ belongs to $Z(u)$ if and only if some node $x \in u$ is among the predecessors of z.

Given an assessment (μ, b), for every terminal node z and every information set $u \in U$ we can calculate the conditional probability of reaching z given that the information set u is reached, as follows:

$$P_u(z|\mu, b) = \begin{cases} \mu(p_m(z)) \cdot \prod_{l=0}^{m-1} b(\lambda(p_l(z))) & \text{if} \quad z \in Z(u) \\ 0 & \text{otherwise} \end{cases}$$

Then, we can define the expected payoff of player i starting from an information set $u \in U$, given an assessment (μ, b) as follows:

$$h_i(b|u, \mu) = \sum_{z \in Z(u)} H_i(z) P_u(z|\mu, b).$$

Definition 2.16. *An assessment (μ, b) is **sequentially rational** if for each player i and each $u \in U_i$,*

$$h_i(b|u, \mu) \geq h_i((b'_i, b_{-i})|u, \mu) \text{ for every } b'_i \in B_i.$$

Definition 2.17. *An assessment (μ, b) is called a **sequential equilibrium** if it is both consistent and sequentially rational.*

3. Games of Simple and Complex Information Structure

In general the information structure of a sequential game may be quite complex, even if we retain the perfect recall assumption. It is worth mentioning here that the perfect recall property is all about player and choice partitions (it imposes restrictions on information sets that belong to the *same* player as well as choices at those information sets). As a consequence, sequential games in which every information set is owned by a different player, regardless of the information partition and ordering of the nodes, is a game of perfect recall.

Our ultimate goal in this paper is to extend the well-known roll-back procedure with respect to the nodes (in case of a perfect information game) or with respect to the subgames (in case of an imperfect information game), known as a backward induction, to the roll-back with respect to the information sets. In order to define it, we need to order the information sets according to their precedence, namely we need a partial order.

Given an extensive game, only the precedence of *nodes* is specified. Let us introduce the concept of the precedence of *information sets* and their simultaneity, the later intuitively meaning that simultaneous information sets are tangled together, or there is no way to tell that one information sets strongly precedes the other.

Definition 3.1. *Given an information set u of the extensive game G, define the set of **first-order successors (or immediate successors)** of u, $Succ_1(u)$, to be the set of all nodes $x \in X$ such that x is in the same information set with an* *immediate successor of* y *(i.e.,* $x, w \in I$ *for some information set* I *and* w *is an immediate successor of* y*) for some* $y \in u$*.*

Definition 3.2. *Given an information set* u*, for every natural number* n *define recursively the set of* k^{th}***-order successors*** *of* u*,* $Succ_k(u)$*, to be the set* $Succ_1(Succ_{k-1}(u))$*. By convention, we let* $Succ_0(u) = \{x \in X : x \in u\}$*.*

Notice that for every finite extensive form game there exists a natural number m such that $Succ_k(u) = \emptyset$ for every $u \in U$ and every $k \geq m$.

Now the notion of a successor of a node easily generalizes to the notion of a successor of an information set. Recall that the set of all successors of a node is the union of its n^{th}-order successors over all $n \in \mathbb{N}$. Same idea is behind the notion of the successors of an information set.

Definition 3.3. *Given an information set* u*, the set of all* ***successors*** *of* u*,* $Succ(u)$*, is the set of nodes*

$$\bigcup_{k=0}^{\infty} Succ_k(u).$$

Now we are in a position to define the relation of precedence of information sets.

Definition 3.4. *Let* I_1 *and* I_2 *be two information sets of the extensive game* G*. We say that* I_1 ***precedes*** I_2 *if* $y \in Succ(I_1)$ *for some* $y \in I_2$*, and write* $I_1 \succeq_p I_2$*.*

As can be seen, the relation $\succeq_p$ is reflexive (since $x \in Succ(x)$ for every $x \in X$) and transitive, but is not antisymmetric (and in fact is not complete). We will call the class of games for which $\succeq_p$ is a partial order games of simple information structure.

Definition 3.5. *Two distinct information sets* I_1 *and* I_2 *are called* ***simultaneous*** *if both* $I_1 \succeq_p I_2$ *and* $I_2 \succeq_p I_1$*.*

Define an equivalence relation $\sim_p$ on the set of information sets of the extensive form game as follows: $I_1 \sim_p I_2$ if $I_1 \succeq_p I_2$ and $I_2 \succeq_p I_1$. A remarkable feature is that passing to the relation $\succeq_p$ modulo $\sim_p$, we obtain a partial order.

Definition 3.6. *An equivalence class of the relation* $\sim_p$ *is called a* ***knot****.*

Let $\mathbb{K}$ denote the set of all knots for the extensive form game G. Notice that since G is a finite game, the set $\mathbb{K}$ is finite, without loss of generality $\mathbb{K} = (K_1, K_2, \cdots, K_m)$. Define a binary relation $\succeq_e$ on $\mathbb{K}$ as follows: $K_1 \succeq_e K_2$ if $u \succeq v$ for some information sets $u \in K_1$ and $v \in K_2$. Then, $K_1 \succeq_e K_2$ can be interpreted as knot K_1 **precedes** knot K_2.

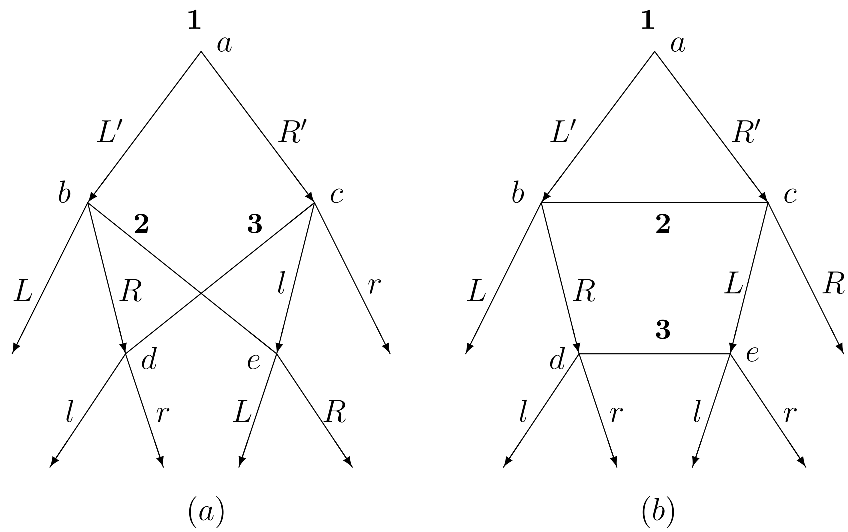

FIGURE 2

Definition 3.7. *An extensive form game G of perfect recall is said to be a game of* ***simple information structure*** *if all its knots are singletons, otherwise it is a game of* ***complex information structure****.*

As an example, consider two sequential games depicted in Figure 2. The game of Figure 2(a) is a game of complex information structure: Here $I_1 = \{a\}$, $I_2 = \{b, e\}$ and $I_3 = \{c, d\}$ are the information sets of players 1, 2 and 3, respectively. Since $b \succ d$ and $c \succ e$, we have both $I_2 \succeq_p I_3$ and $I_3 \succeq_p I_2$, so that $I_2 \sim_p I_3$. On the other hand, $I_1 \succ_p I_2$ and $I_1 \succ_p I_3$. Therefore this game has two knots: $K_1 = I_1$ and $K_2 = \{I_2, I_3\}$.

On the other hand, the game of Figure 2(b) is a game of simple information structure: here all three knots $K_1 = I_1 = \{a\}$, $K_2 = I_2 = \{b, c\}$ and $K_3 = I_3 = \{d, e\}$ are singletons.

Notice that for the games of simple information structure the relation $\succeq_p$ modulo $\sim_p$ is the relation $\succeq_p$ itself. Intuitively, in a game of simple information structure information sets are aligned one after another, so that relation $\succeq_p$ is a partial order. This suggests a straightforward generalization of the backward induction process for the games of simple information structure.

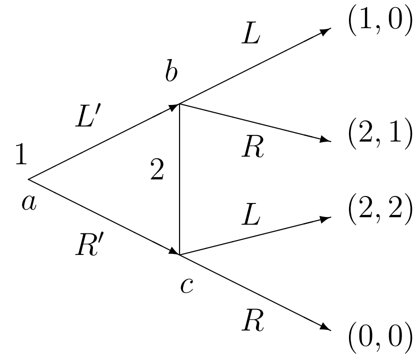

FIGURE 3

4. EXAMPLES OF APPLYING THE ROLL-BACK PROCEDURE TO FIND ALL SEQUENTIAL EQUILIBRIA OF EXTENSIVE GAMES WITH PERFECT RECALL

Example 4.1. Consider the game depicted in Figure 3. Let us solve this game backwards, starting from the information set of player 2. Fix a vector of beliefs $(\mu(b), \mu(c))$ at the information set of player 2. Consider the following cases.

If $\mu(b) > \frac{2}{3}$, then the best response of player 2 is R. Truncate the game tree according to this optimal choice, assigning the payoff vector $(2, 1)$ to node b, and the payoff vector $(0, 0)$ to node c. Then at a player 1 selects L' with probability 1, generating beliefs $(1, 0)$ at the information set of player 2, which are consistent with the initial assumption $\mu(b) > \frac{2}{3}$. Hence, the assessment $((p^*_{L'}, p^*_{R'}), (\mu^*(b), \mu^*(c)), (p^*_L, p^*_R)) = ((1, 0), (1, 0), (0, 1))$ is a sequential equilibrium of the given game.

Assume $\mu(b) < \frac{2}{3}$, then player 2 optimally selects L. Truncate the game tree according to this optimal choice, assigning the payoff vector $(1, 0)$ to node b, and the payoff vector $(2, 2)$ to node c. Then player 1 chooses R' with probability 1 at his decision node a, which generates beliefs $(0, 1)$ at the information set of player 2. This is consistent with the initial assumption $\mu(b) < \frac{2}{3}$. Hence, the assessment $(b^*_1, (\mu^*(b), \mu^*(c)^*), b^*_2) = ((0, 1), (0, 1), (1, 0))$ is also a sequential equilibrium.

Finally, consider the case $\mu(b) = \frac{2}{3}$, then the whole 1-simplex $[L, R]$ is a best response of player 2. The only strategy profile of player 1 that supports these beliefs is $(p_{L'}, p_{R'}) = (\frac{2}{3}, \frac{1}{3})$. Since it is in the interior of the strategy space B_1 of player 1, then player 1 must be indifferent between his pure strategies L' and R'. In other words, the strategy profile (p_L, p_R) of player 2 must satisfy

$$p_L + 2p_R = 2p_L,$$

or $(p_L, p_R) = (\frac{2}{3}, \frac{1}{3})$. Hence, the strategy profile $((\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}))$ is the only remaining sequential equilibrium.

An important issue needs to be highlighted here. Given an extensive-form representation of any normal form game, the set of sequential equilibria for this game coincides with the set of Nash equilibria. This is due to the fact that for such games the belief correspondence is a function, i.e., for every behavior strategy profile there exists a unique belief assignment consistent with it. This gives an unexpected flavor to the generalized backward induction procedure, employed to find all sequential equilibria: it is equivalent to solving a fixed point problem!

Example 4.2. Consider the "Selten's horse" ([16], p.870) depicted in Figure 4. Denote a generic assessment for the game by

$$(b_1, b_2, b_3, (\mu(y), \mu(z))) = ((p_A, p_D), (p_a, p_d), (p_l, p_r), (\mu(y), \mu(z))).$$

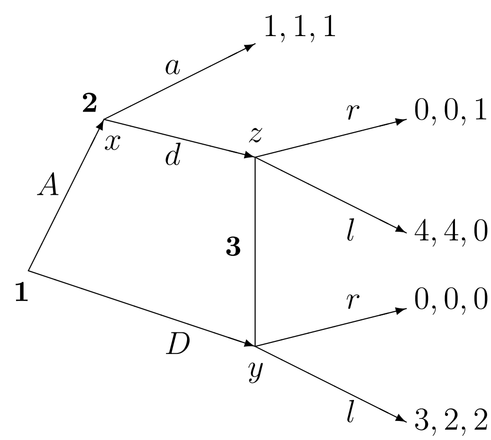

FIGURE 4

Again, let us solve the game backwards starting from the information set of player 3. Consider three possible cases:

(1) If $\mu(y) > \frac{1}{3}$, the optimal choice of player 3 at his information set $\{y, z\}$ is playing l with probability one. Truncate the game tree according to player 3's best response, assigning the payoff vector $(3, 2, 2)$ to node y, and the payoff vector $(4, 4, 0)$ to node z..

Given this truncation, player 2 at his decision node selects d with probability one. Truncate the reduced game tree, assigning the payoff $(4, 4, 0)$ to node x. Finally, player 1 faces the problem of randomizing between A, which brings him payoff of 4, and D resulting in payoff 3. Then, player 1 selects A with probability one. Notice that the obtained behavior strategy profile generates beliefs $(0, 1)$ over the information set of player 3. However, this is inconsistent with our initial assumption that $\mu(y) > \frac{1}{3}$.

(2) Suppose $\mu(y) < \frac{1}{3}$, then the best response of player 3 is to play r with probability one. Given the optimal choice of player 3, truncate the game tree, assigning the payoff vector $(0, 0, 0)$ to node y, and the payoff vector $(0, 0, 1)$ to node z.

Given this truncation, player 2 at his decision node selects a with probability one, for he prefers a payoff of 1, resulting from choosing a, to 0, resulting from d. Truncate the reduced game tree, assigning the payoff $(1, 1, 1)$ to node x. Now player 1 is the only player and chooses A with probability one, which gives him payoff 1.

We thus obtain behavior strategy profile $b^* = (b_1^*, b_2^*, b_3^*) = ((1, 0), (1, 0), (0, 1))$. Notice that any beliefs are consistent with b^*, however only beliefs allocating probability at least $\frac{2}{3}$ to node z satisfy our initial requirement (that is, support the choice r of player 3. Therefore, any assessment of the form

$$((1, 0), (1, 0), (0, 1), (\alpha, 1 - \alpha))$$

with $0 \leq \alpha < \frac{1}{3}$ is a sequential equilibrium.

(3) Finally, suppose $\mu(y) = \frac{1}{3}$, then any behavior strategy $b_3 \in B_3$ of player 3 (that is, any randomization between his choices l and r) is a best response. Denote by $p(l)$ and $p(r)$ probabilities with which player 3 selects l and r, respectively. Truncate the game tree, assigning the payoff vector $(3p(l), 2p(l), 2p(l))$ to node y, and payoff $(4p(l), 4p(l), 1 - p(l))$ to node z. Consider the following subcases:

(a) Suppose player 2 chooses strategy d with probability one, which is sequentially rational if and only if player 3 plays l with probability at least $\frac{1}{4}$, that is, $p(l) \geq \frac{1}{4}$. This is so because playing the pure strategy d brings player 2 payoff at least 1 (which is a payoff from playing a) if and only if $p(l) \geq \frac{1}{4}$. Truncate the game tree once again, assigning payoff $(4p(l), 4p(l), 1 - p(l))$ to node x. But then player 1 faces a problem of randomizing between his pure strategies A, which brings him a payoff $4p(l)$, and D, which brings him a payoff $3p(l)$. Clearly, for any $p(L) > 0$ he would choose A with probability one (notice that case $p(l) = 0$ was considered earlier, see Case 2). However the obtained strategy profile induces beliefs $\mu(y) = 0$, which contradicts the earlier requirement $\mu(y) = \frac{1}{3}$.

(b) Suppose player 2 is playing some nondegenerate convex combination of a and d, which is sequentially rational if and only if $p(l) = \frac{1}{4}$. Notice that under this condition, for any strategy of player 2, the payoff of player 1 starting from node x is 1. Truncate the game tree once again, assigning the payoff vector $(1, \alpha, \beta)$ to node x (notice that the expected payoffs of players 2 and 3 are irrelevant for an optimal choice of player 1, so we denote them by parameters α and β). Clearly, given $p(l) = \frac{1}{4}$, player 1 will choose A with probability one. Again, this induces beliefs $mu(y) = 0$, which contradicts the initial requirement $\mu(y) = \frac{1}{3}$.

(c) Suppose player 2 chooses strategy a with probability one, which is sequentially rational if and only if $p(l) \leq \frac{1}{4}$. Truncate the reduced game tree, assigning payoff $(1, 1, 1)$ to node x. Given $p(l) \leq \frac{1}{4}$, after truncation strategy D brings player 1 payoff at most $\frac{3}{4}$, while strategy A results in payoff 1. Therefore player 1 will choose A with probability one. Any belief system is consistent with this strategy profile, however the initial requirement imposes $\mu(y) = \frac{1}{3}$. Hence, any assessment $((1, 0), (1, 0), (\gamma, 1 - \gamma), (\frac{1}{3}, \frac{2}{3}))$, where $\gamma \leq \frac{1}{4}$, is sequentially rational.

There is an efficient procedure, which is suitable to find all generalized backward induction assessments, and in many cases can be performed easily (as in the above examples). The next section formalizes the generalized backward induction procedure as a finite sequence of steps, and nonemptiness of the resulting set of assessments is secured by the generalization of Eilenberg-Montgomery fixed-point theorem [17].

5. Generalized Backward Induction: Definition, and Direct Proof of the Existence of Sequential Equilibria

Let G be an extensive form game of simple information structure. Then, the binary relation $\succeq_p$ on the set of information sets U is a partial order. Let $\mathcal{F}_0$ be the set of all minimal elements of U with respect to the partial order $\succeq_p$, that is, the set of all information sets u such that there is no information set v with the property $u \succ_p v$.

We claim that $\mathcal{F}_0$ is nonempty. Indeed, notice that since G is a finite game, the set U is finite and partially ordered by $\succeq_p$. Also U has finitely many chains, call them $\mathcal{C}_1, \cdots, \mathcal{C}_q$, and every chain has finitely many elements. Hence every chain C_i has a (unique) minimal element u_i. Then $\mathcal{F}_0 = \{u_1, \cdots, u_q\} \neq \emptyset$, q.e.d.

Call children of $\mathcal{F}_0$ the set of children of all nodes from any information set of $\mathcal{F}_0$, and denote the set of children of $\mathcal{F}_0$ by $\mathcal{N}_0$. The following result is crucial for defining the generalized backward induction, which fails for games of complex information structure.

Lemma 5.1. *Let G be a sequential game of simple information structure. Then all children of $\mathcal{F}_0$ are terminal nodes.*

Proof. Assume G is a sequential game of simple information structure. It was shown above that $\mathcal{F}_0 \neq \emptyset$. Suppose by contradiction there exists an information set $u \in \mathcal{F}_0$, node $x \in u$ and a child y of x such that y is a non-terminal node (this implies y is a decision node). Without loss of generality, let v be the information set containing y. Notice that u is different from v by definition of an information set (since we have $x \succ y$).

Also, since $x \succ y$, we have $u \succeq_p v$. We claim that $u \not\sim_p v$. Indeed, if $u \sim_p v$, then the order $\succeq_p$ is not antisymmetric, and hence not a partial order, contradiction. Then we have both $u \succeq_p v$ and $u \not\sim_p v$, which implies $u \succ_p v$. But then u is not a minimal element with respect to $\succeq_p$, contradiction. This completes the proof. ■

Fix a system of beliefs $\mu \in M$ and proceed recursively as follows. Denote by $\mu_{\mathcal{F}_0}$ the restriction of μ to the set of information sets $\mathcal{F}_0$ (note that $\mathcal{F}_0 \neq \emptyset$). Given $\mu_{\mathcal{F}_0}$, the set of best responses, or sequentially rational behavior strategy profiles $b^*_{\mathcal{F}_0}$ (call it $\Lambda_{\mathcal{F}_0}(\mu)$) is nonempty, convex and compact subset of $B_{\mathcal{F}_0}$.

Fix $b^*_{\mathcal{F}_0} \in \Lambda_{\mathcal{F}_0}(\mu)$. Truncate the extensive form Ξ, deleting all nodes in $\mathcal{N}_0$, assigning to each node in $\mathcal{F}_0$ the expected payoff vector generated by $b^*_{\mathcal{F}_0}$ (we are using here Lemma 5.1, which guarantees that all nodes in $\mathcal{N}_0$ are terminal nodes). We thus obtained a new sequential game, call it G_1, with the corresponding extensive form Ξ_1.

It can be easily verified that $\succeq_p$ restricted to Ξ_1 is a partial order. Denote by $\mathcal{F}_1$ the set of minimal elements among the information sets of G_1 with respect to the partial order $\succeq_p$ restricted to Ξ_1. By the earlier argument, the set $\mathcal{F}_1$ is nonempty. Repeat the above steps, applied to the game G_1.

Since Ξ has finitely many nodes, at some point the process will stop. Thus, for a given system of beliefs μ we can recursively construct a finite sequence of games $G_1, \cdots, G_r$.

Definition 5.2. *Given an extensive form game G of simple information structure, the process of constructing the sequence of games $G_1, \cdots, G_r$ for a given beliefs system μ described above is called a* ***generalized backward induction process****. Let b^* be a behavior strategy profile generated by this process, then if the corresponding system of beliefs μ is consistent with b^* (i.e., μ is in $\phi(b^*)$), call the assessment (μ, b^*) a* ***backward induction assessment****.*

Notice that since $\succeq_p$ is a partial order, the generalized backward induction process will stop at the root information set I_W. This is due to the fact that I_W is the unique maximal element of U with respect to $\succeq_p$ for the game G.

Lemma 5.3. *Let G be an extensive form game of simple information structure. Then I_W, the root information set, is the unique maximal element of U with respect to the partial order $\succeq_p$.*

Proof. First, let us show that I_W is a maximal element. Suppose by contradiction there exists $u \in U$ such that $u \succ_p I_W$. This implies there exist nodes $x \in u$, $w \in I_W$ such that $x \succ w$, that is, x is a predecessor of w. But this is impossible since $w \in I_W$ is a root.

Next, let us show that there is no maximal element other than I_W. Suppose by contradiction there exists an information set v different from I_W, such that v is a maximal element of U with respect to $\succeq_p$. Fix an element $y \in v$. By assumption y is a non-root node, hence there exists a root node $w \in I_W$ such that there is a unique path from w to y. But this implies $w \succ y$, hence $I_W \succ_p v$. This contradicts our assumption that v is a maximal element. This establishes that I_W is the unique maximal element. ■

It is important to emphasize that for a given $\mu \in M$, the generalized backward induction process may yield infinitely, in fact uncountably many behavior strategy profiles. This is because the set of best responses at any step may be uncountable. Also, in order to find all backward induction assessments, one needs to perform the generalized backward induction for every system of beliefs $\mu \in M$.

Next, we need to show that the backward induction assessments are precisely the sequential equilibria of a given sequential game G, and the set of backward induction assessments is nonempty. The first result (Theorem 5.5) can be established using mathematical induction. Its proof also relies on the following lemma, which is an easy consequence of the continuity of the dot product.

Lemma 5.4. *Let G be an extensive form game, and information sets $u, v \in U$ be such that every node in v has a predecessor among nodes of u. Let (μ, b) be a consistent assessment. If $P_u(y|\mu, b) \neq 0$ for some $y \in v$, then for every $x \in v$*

$$\mu(x) = \frac{P_u(x|\mu, b)}{\sum_{y \in v} P_u(y|\mu, b)}.$$

Theorem 5.5. *Let G be an extensive form game of simple information structure. Then an assessment (μ^*, b^*) is a backward induction assessment if and only if it is a sequential equilibrium of G.*

Proof. ($\Rightarrow$) Assume (μ^*, b^*) is a backward induction assessment. Then the system of beliefs μ^* is consistent with b^* by definition. It remains to show that (μ^*, b^*) is sequentially rational. This can be established using the method of mathematical induction. WIthout loss of generality, suppose the backward induction process stops at $\mathcal{F}_r$ for some $r \in \mathbb{N}$, i.e., the root information set I_W is in $\mathcal{F}_r$.

Fix a natural number $0 \leq k < r$. By induction hypothesis, assume (μ^*, b^*) is sequentially rational starting from any information set in $\mathcal{F}_j$, for all $0 \leq j \leq k$. We need to show that (μ^*, b^*) is sequentially rational starting from any information set in $\mathcal{F}_{k+1}$. So, fix an information set $u \in \mathcal{F}_{k+1}$, which is owned by player i, without loss of generality.

If player i does not own any information set among $Succ(u)$, we are done, since b^*_{iu} is a best response of player i at u at the $(k+2)^{th}$ step of the generalized backward induction process.

Assume player i owns some information set in $Succ(u)$. Suppose by contradiction player i wants to deviate, starting from u, at a nonempty set V_i of information sets. Consider two possible cases:

(1) $V_i \cap \mathcal{F}_{k+1} = \emptyset$. Notice that the set V_i is finite and partially ordered by $\succeq_p$. Fix a chain $\mathcal{C}$ in V_i. Since it is finite and totally ordered, it has a unique maximal element, call it v. But this implies that player I wants to deviate at v, which contradicts our induction hypothesis, since $v \in \mathcal{F}_j$ for some $0 \leq j \leq k$.

(2) $V_i \cap \mathcal{F}_{k+1} \neq \emptyset$, which means $V_i \cap \mathcal{F}_{k+1} = u$. Then:
   (a) If $V_i \backslash u = \emptyset$, we arrive at a contradiction, since b^*_{iu} is a best response of player i at u at the $(k+2)^{th}$ step of the generalized backward induction process.
   (b) Suppose $V_i \backslash u \neq \emptyset$. Without loss of generality let $\tilde{b} \in B$ be the behavior strategy profile such that $\tilde{b}_m = b_j$ for all $m \neq i$, and $h_i(\tilde{b}|u, \mu) > h_i(b|u, \mu)$. The perfect recall assumption together with Lemma 5.4 implies that for every choice $s_j \in C_u$ of player i at u,

$$h_i((s_j, \tilde{b}\backslash s_j)|u, \mu) > h_i((s_j, b\backslash s_j)|u, \mu).$$

   But then, deviating to $\tilde{b}_i$ from b_i by player i is equivalent to deviating at the information set u only. As before, this is a contradiction, since b^*_{iu} is a best response of player i at u at the $(k+2)^{th}$ step of the generalized backward induction process.

This shows (μ^*, b^*) is sequentially rational starting from any information set in $\mathcal{F}_{k+1}$. Hence by mathematical induction (μ^*, b^*) is sequentially rational.

($\Leftarrow$) Assume (μ^*, b^*) is a sequential equilibrium. Hence $b^*_{\mathcal{F}_0}$ is optimal starting from any information set in $\mathcal{F}_0$ given $\mu^*_{\mathcal{F}_0}$. Therefore $b^*_{\mathcal{F}_0}$ is selected on the first step of the generalized backward induction.

Assume by induction hypothesis that $b^*_{\mathcal{F}_k}$ is selected on the $(k+1)^{th}$ step of backward induction, after truncating the extensive form according to $(b^*_{\mathcal{F}_l})_{l=0,\cdots,k-1}$. Suppose $b^*_{\mathcal{F}_{k+1}}$ is not optimal starting from some information set $u \in \mathcal{F}_{k+1}$, after truncating the extensive form according to $(b^*_{\mathcal{F}_l})_{l=0,\cdots,k}$. But this implies the owner of u has an incentive to deviate from b^*, starting at u, hence (μ^*, b^*) is not sequentially rational, which is a contradiction. We are done by mathematical induction.

■

Now we are ready to provide a direct proof of the existence of sequential equilibria in games of simple information structure. We do this by showing that the set of backward induction assessments is nonempty. An indirect proof of the existence of sequential equilibria in extensive games with perfect recall, given by Kreps and Wilson, can be found in [8].

Define a correspondence $\tau : M \times B \twoheadrightarrow M \times B$ as follows: given $(\mu^*, b^*) \in M \times B$, let

$$\tau(\mu^*, b^*) = \tilde{M} \times \tilde{B},$$

where $\tilde{M}$ is the set of all beliefs systems consistent with b^*, and

$\tilde{B} = \tilde{B}_{\mathcal{F}_r} \times \cdots \times \tilde{B}_{\mathcal{F}_0}$, such that for each $j = 0, \cdots, r$, $\tilde{B}_{\mathcal{F}_j}$ is the set of best replies at $\mathcal{F}_j$ given ρ, $\mu^*_{\mathcal{F}_j}$ and the truncation according to $(b^*_{\mathcal{F}_{j-1}}, \cdots, b^*_{\mathcal{F}_0})$.

It follows immediately from the definition of a backward induction assessment that $(\mu, b) \in M \times B$ is a backward induction assessment if and only if (μ, b) is a fixed point of τ.

It remains to show that τ has a fixed point. Note that τ has a problematic boundary behavior: if $b \in \partial B$, then the set of beliefs consistent with b is not necessarily convex or even acyclic. However, τ is nicely behaved on the interior of $M \times B$. This suggests applying the generalization of Eilenberg-Montgomery fixed point theorem as appeared in [17].

Lemma 5.6. *The correspondence τ is convex-valued on the interior of $M \times B$.*

Proof. Fix $(\mu, b) \in M^\circ \times B^\circ$, without loss of generality $\tau(\mu, b) = \tilde{M} \times \tilde{B}$. Then $\tilde{M}$ is a singleton, hence is a convex subset of M.

For each $j = 0, \cdots, r$, $\tilde{B}_{\mathcal{F}_j}$ is the set of best replies at $\mathcal{F}_j$ given ρ, $\mu_{\mathcal{F}_j}$ and the truncation according to $(b_{\mathcal{F}_{j-1}}, \cdots, b_{\mathcal{F}_0})$. Therefore $\tilde{B}_{\mathcal{F}_j}$ is nonempty and convex for each $j = 0, \cdots, r$. This implies $\tilde{B} = \tilde{B}_{\mathcal{F}_r} \times \cdots \times \tilde{B}_{\mathcal{F}_0}$ is a convex subset of B. Consequently, $\tau(\mu, b) = \tilde{M} \times \tilde{B}$ is convex. ■

Lemma 5.7. *The correspondence τ is nonempty-valued and closed.*

Proof. (i) First let us show that τ is closed. Fix a sequence $(\mu^k, b^k) \subseteq M \times B$ such that $(\mu^k, b^k) \to (\mu^*, b^*) \in M \times B$ as $k \to \infty$. Let $(\tilde{\mu}^k, \tilde{b}^k) \in \tau(\mu^k, b^k)$ for each $k \in \mathbb{N}$ such that $(\tilde{\mu}^k, \tilde{b}^k) \to (\tilde{\mu}^*, \tilde{b}^*)$. We need to show that $(\tilde{\mu}^*, \tilde{b}^*) \in \tau(\mu^*, b^*)$.

Without loss of generality $\tau(\mu^k, b^k) = M^k \times B^k$ for each $k \in \mathbb{N}$ and $\tau(\mu^*, b^*) = M^* \times B^*$. By the closedness of the belief correspondence ϕ, $\tilde{\mu}^* \in M^*$.

Fix $j \in \{0, \cdots, r\}$, and notice that the payoff of the player decisive at each information set $u \in \mathcal{F}_j$, given ρ, $\mu_{\mathcal{F}_j}$ and the truncation according to $(b_{\mathcal{F}_{j-1}}, \cdots, b_{\mathcal{F}_0})$, is jointly continuous in μ and b. Therefore $b^* \in B^*$. This shows $(\tilde{\mu}^*, \tilde{b}^*) \in \tau(\mu^*, b^*)$, hence τ has a closed graph.

(ii) By the closedness of τ and compactness of $M \times B$ it suffices to show that τ is nonempty-valued on the interior of $M \times B$. Fix $(\mu, b) \in M^\circ \times B^\circ$, and follow exactly the lines of the proof of Lemma 5.6 to establish that $\tau(\mu, b) \neq \emptyset$. ■

Lemmas 5.6 and 5.7 imply that τ satisfies the hypotheses of [17, Theorem 3.10, p. 5]. Also, $M \times B$ satisfies the hypotheses of [17, Theorem 3.10, p. 5], since M and B are nonempty, convex and compact subsets of some Euclidean space. Therefore, we get the following existence result.

Theorem 5.8. *τ has a fixed point over $M \times B$, and consequently, the set of backward induction assessments for an extensive game of simple information structure is nonempty.*

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis: A Hitchhikers Guide*, 3rd Edition, Springer–Verlag, New York & London, 2006.

[2] C. D. Aliprantis and S. K. Chakrabarti, *Games and Decision Making*, Oxford University Press, New York & London, 2000.

[3] C. D. Aliprantis, On the backward unduction method, *Economics Letters* **64** (1999), 125–131.

[4] I. Cho and D. M. Kreps, Signaling games and stable equilibria, *Quarterly Journal of Economics* **2** (1987), 179–221.

[5] D. Fudenberg and E. Maskin, The folk theorem in repeated games with discounting or with incomplete information, *Econometrica* **54** (1986), 533–554.

[6] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, Cambridge, MA, 1991.

[7] E. Klein and A. C. Thompson, *Theory of Correspondences*, John Wiley & Sons, New York, 1984.

[8] D. M. Kreps and R. Wilson, Sequential Equilibria, *Econometrica* **50** (1982), 863–894.

[9] G. J. Mailath and L. Samuelson, *Repeated Games and Reputations*, Oxford University Press, Oxford & New York, 2006.

[10] A. Mas-Colell, M. D. Whinston, and J. R. Green, *Microeconomic Theory*, Oxford University Press, New York & Oxford, 1995.

[11] R. Myerson, *Game Theory: Analysis of Conflict*, Harvard University Press, Cambridge, MA, 1997.

[12] J. Nash, Equilibrium points in N-person games, *Proceedings of the National Academy of Sciences* **36** (1950), 48–49.

[13] J. Nash, Non-cooperative games, *Annals of Mathematics* **54(2)** (1951), 286–295.

[14] M. J. Osborne, *An Introduction to Game Theory*, Oxford University Press, Oxford & New York, 2004.

[15] E. Rasmusen, *Games and Information: an Introduction to Game Theory*, 4th Edition, Blackwell Publishers, Oxford, 2007.

[16] R. Selten, Re-examination of the perfectness concept for the equilibrium points in extensive games, *International Journal of Game Theory* **4** (1975), 24–55.

[17] I. Topolyan, Existence of perfect equilibria: a direct proof, *Krannert Working Paper #1226* (2009).