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# MPCC: Critical Point Theory

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# MPCC: Critical Point Theory

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**Abstract**

We study mathematical programs with complementarity constraints (MPCC) from a topological point of view. Special focus will be on C-stationary points. Under the Linear Independence Constraint Qualification (LICQ) we derive an equivariant Morse Lemma at nondegenerate C-stationary points. Then, two basic theorems from Morse Theory (deformation theorem and cell-attachment theorem) are proved. Outside the C-stationary point set, continuous deformation of lower level sets can be performed. As a consequence, the topological data (such as the number of connected components) then remain invariant. However, when passing a C-stationary level, the topology of the lower level set changes via the attachment of a $q$-dimensional cell. The dimension $q$ equals the stationary C-index of the (nondegenerate) C-stationary point. The stationary C-index depends on both the restricted Hessian of the Lagrangian and the Lagrange multipliers related to bi-active complementarity constraints. Finally, some relations with other stationarity concepts, such as W-, A-, M-, S- and B-stationarity, are discussed.

**Keywords**: Mathematical programs with complementarity constraints, C-stationarity, Linear Independence Constraint Qualification, Stationary C-index, Morse Theory.

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# 1 Introduction

We consider the following mathematical programming problem with complementarity constraints (MPCC):

$$\text{MPCC:} \quad \min f(x) \text{ s.t. } x \in M[h, g, F_1, F_2] \tag{1}$$

with

$$\begin{aligned} M[h, g, F_1, F_2] := \{x \in \mathbb{R}^n \mid \; & F_{1,m}(x) \geq 0, F_{2,m}(x) \geq 0, \\ & F_{1,m}(x)F_{2,m}(x) = 0, \, m = 1, \ldots, k, \\ & h_i(x) = 0, \, i \in I, \, g_j(x) \geq 0, \, j \in J\}, \end{aligned}$$

where $h := (h_i, \, i \in I)^T \in C^2(\mathbb{R}^n, \mathbb{R}^{|I|})$, $g := (g_j, \, j \in J)^T \in C^2(\mathbb{R}^n, \mathbb{R}^{|J|})$, $F_1 := (F_{1,i}, \, i = 1, \ldots, k)^T, F_2 := (F_{2,i}, \, i = 1, \ldots, k)^T \in C^2(\mathbb{R}^n, \mathbb{R}^k)$, $f \in C^2(\mathbb{R}^n, \mathbb{R})$, $k+|I| \leq n$, $|J| < \infty$. For simplicity, we write $M$ for $M[h, g, F_1, F_2]$, if needed.

The goal of this paper is the study of MPCC from a topological point of view. It turns out that the concept of C-stationarity is the adequate stationarity concept. To this aim we study the behaviour of the topological properties of lower level sets

$$M^a := \{x \in M \mid f(x) \leq a\}$$

as the level $a \in \mathbb{R}$ varies. In particular, within this context, we present two basic theorems form Morse Theory (cf. [7, 10]). First, we show that, for $a < b$, the set $M^a$ is a strong deformation retract of $M^b$ if the (compact) set

$$M_a^b := \{x \in M \mid a \leq f(x) \leq b\}$$

does not contain C-stationary points (see Theorem 3.2(a)). As a consequence, the homotopy type of the lower level sets $M^a$ and $M^b$ are equal. This means that the connectedness structure of the lower level sets does not change when passing from level $a$ to level $b$. In particular, the number of connected (path)components remains invariant. Second, if $M_a^b$ contains exactly one (nondegenerate) C-stationary point, then $M^b$ is shown to be homotopy equivalent to $M^a$ with a $q$-cell attached (see Theorem 3.2(b)). Here, the dimension $q$ is the socalled C-index. It depends on both the restricted Hessian of the Lagrangian and the Lagrange multipliers related to bi-active complementarity constraints. The latter fact is the main difference with respect to the well-known case where feasible set is described only by equality and finitely many inequality constraints (cf. [7]).

A global interpretation of our results is the following. Suppose that the feasible set is compact and connected, that a suitable constraint qualification holds (e.g. LICQ, see later on), and that all C-stationary points are non-degenerate with pairwise different functional values. Then, passing a level corresponding to a local minimizer, a connected component of the lower level set is created. Different components can only be connected by attaching 1-cells. This shows the existence of at least $(k-1)$ C-stationary points with C-index equal to one, where $k$ is the number of local minimizers, see also [3, 7].

For the proof of the above mentioned results we locally describe the MPCC feasible set under the Linear Independence Constraint Qualification (see Lemma 2.1). Moreover, an equivariant Morse Lemma for MPCC is derived in order to obtain suitable normal forms for the objective function at C-stationary points (see Theorem 3.1).

We would like to refer to some related papers. In [14] the concept of a non-degenerate feasible point for MPCC is introduced. Some genericity results are obtained. In [12] the concepts of a non-degenerate C-stationary point and its stationary C-index are introduced for quadratic programs with complementarity constraints (QPCC). The generic structure of the C-stationary point set for non-parametric and one-parametric QPCCs is discussed and some homotopy methods for QPCC are developed.

The article is organized as follows. In Section 2 we provide notations and auxiliary results which will be used later. Section 3 contains the Morse Lemma for MPCC as well as the proofs of the deformation- and cell-attaching theorem. In Section 4 some relations to other stationarity concepts (such as W-, A-, M-, S- and B-stationarity) are discussed.

Our notation is standard. The $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$, its nonnegative orthant by

$$\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_i \geq 0, \, i = 1, \ldots, n\}.$$

With this notation, $\partial \mathbb{H}^2$ represents the basic complementarity relation:

$$u \geq 0, \, v \geq 0, \, u \cdot v = 0.$$

Given an arbitrary set $K \subset \mathbb{R}^n$, $\partial K$ denotes the topological boundary of $K$. Given a differentiable function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $DF$ denotes its Jacobi-matrix. Given a differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $Df$ denotes its gradient as a row vector and $D^T f$ stands for the transposed vector.

## 2 Notations and Auxiliary Results

Given $\bar{x} \in M$, we define the following index sets:

$$J_0(\bar{x}) := \{j \in J \mid g_j(\bar{x}) = 0\},$$

$$\begin{aligned}\alpha(\bar{x}) &:= \{m \in \{1, \dots k\} \mid F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) > 0\},\\ \beta(\bar{x}) &:= \{m \in \{1, \dots k\} \mid F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) = 0\},\\ \gamma(\bar{x}) &:= \{m \in \{1, \dots k\} \mid F_{1,m}(\bar{x}) > 0, F_{2,m}(\bar{x}) = 0\}.\end{aligned}$$

We call $J_0(\bar{x})$ the active inequality index set and $\beta(\bar{x})$ the bi-active index set at $\bar{x}$.

Without loss of generality (w.l.o.g.), we assume throughout the whole article that at the particular point of interest $\bar{x} \in M$ it holds:

$$J_0(\bar{x}) = \{1, \dots, |J_0(\bar{x})|\},\ \alpha(\bar{x}) = \{1, \dots, |\alpha(\bar{x})|\},$$

$$\gamma(\bar{x}) = \{|\alpha(\bar{x})| + 1, \dots, |\alpha(\bar{x})| + |\gamma(\bar{x})|\}.$$

We put $s := |I| + |\alpha(\bar{x})| + |\gamma(\bar{x})|$, $q := s + |J_0(\bar{x})|$, $p := n - q - 2|\beta(\bar{x})|$.

Further, we recall the well-known Linear Independence Constraint Qualification (LICQ) for MPCC (e.g. [13]), which is said to hold at $\bar{x} \in M$ if the set of vectors

$$\begin{aligned}&\{D^T h_i(\bar{x}),\ i \in I, D^T F_{1,m_\alpha}(\bar{x}),\ m_\alpha \in \alpha(\bar{x}),\ D^T F_{2,m_\gamma}(\bar{x}),\ m_\gamma \in \gamma(\bar{x}),\\ &D^T g_j(\bar{x}),\ j \in J_0(\bar{x}),\ D^T F_{1,m_\beta}(\bar{x}),\ D^T F_{2,m_\beta}(\bar{x}),\ m_\beta \in \beta(\bar{x})\}\end{aligned}$$

is linearly independent.

**Definition 2.1 (C-stationary point, cf. [1, 13])** *A point $\bar{x} \in M$ is called Clarke stationary (C-stationary) for MPCC if there exist real numbers $\bar{\lambda}_i$, $i \in$*

*$I$, $\bar{\varrho}_{m_\alpha}$, $m_\alpha \in \alpha(\bar{x})$, $\bar{\vartheta}_{m_\gamma}$, $m_\gamma \in \gamma(\bar{x})$, $\bar{\mu}_j$, $j \in J_0(\bar{x})$, $\bar{\sigma}_{1,m_\beta}$, $\bar{\sigma}_{2,m_\beta}$, $m_\beta \in \beta(\bar{x})$, (Lagrange multipliers), such that:*

$$Df(\bar{x}) = \sum_{i \in I} \bar{\lambda}_i Dh_i(\bar{x}) + \sum_{m_\alpha \in \alpha(\bar{x})} \bar{\varrho}_{m_\alpha} DF_{1,m_\alpha}(\bar{x}) + \sum_{m_\gamma \in \gamma(\bar{x})} \bar{\vartheta}_{m_\gamma} DF_{2,m_\gamma}(\bar{x})$$

$$+ \sum_{j \in J_0(\bar{x})} \bar{\mu}_j Dg_j(\bar{x}) + \sum_{m_\beta \in \beta(\bar{x})} \left( \bar{\sigma}_{1,m_\beta} DF_{1,m_\beta}(\bar{x}) + \bar{\sigma}_{2,m_\beta} DF_{2,m_\beta}(\bar{x}) \right), \quad (2)$$

$$\bar{\mu}_j \geq 0 \text{ for all } j \in J_0(\bar{x}), \quad (3)$$

$$\bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} \geq 0 \text{ for all } m_\beta \in \beta(\bar{x}). \quad (4)$$

In the case where LICQ holds at $\bar{x} \in M$, the Lagrange multipliers in (2) are uniquely determined.

Given a C-stationary point $\bar{x} \in M$ for MPCC, we set:

$$\begin{aligned} M(\bar{x}) := \{x \in \mathbb{R}^n \mid \quad & h_i(x) = 0,\, i \in I, F_{1,m_\alpha}(x) = 0,\, m_\alpha \in \alpha(\bar{x}), \\ & F_{2,m_\gamma}(x) = 0,\, m_\gamma \in \gamma(\bar{x}),\, g_j(x) = 0,\, j \in J_0(\bar{x}), \\ & F_{1,m_\beta}(x) = 0,\, F_{2,m_\beta}(x) = 0,\, m_\beta \in \beta(\bar{x})\}. \end{aligned}$$

Obviously, $M(\bar{x}) \subset M$ and, in the case where LICQ holds at $\bar{x}$, $M(\bar{x})$ is locally a $p$-dimensional $C^2$-manifold.

**Definition 2.2 (Nondegenerate C-stationary point, cf. [12, 14])** *A C-stationary point $\bar{x} \in M$ with Lagrange multipliers as in Definition 2.1 is called nondegenerate if the following conditions are satisfied:*

*ND1: LICQ holds at $\bar{x}$,*

*ND2: $\bar{\mu}_j > 0$ for all $j \in J_0(\bar{x})$,*

*ND3: $D^2L(\bar{x}) \mid_{T_{\bar{x}}M(\bar{x})}$ is nonsingular,*

*ND4: $\bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} > 0$ for all $m_\beta \in \beta(\bar{x})$.*

Here, the matrix $D^2L$ stands for the Hessian of the Lagrange function $L$,

$$L(x) := f(x) - \sum_{i \in I} \bar{\lambda}_i h_i(x) - \sum_{m_\alpha \in \alpha(\bar{x})} \bar{\varrho}_{m_\alpha} F_{1,m_\alpha}(x) - \sum_{m_\gamma \in \gamma(\bar{x})} \bar{\vartheta}_{m_\gamma} F_{2,m_\gamma}(x)$$

$$- \sum_{j \in J_0(\bar{x})} \bar{\mu}_j g_j(x) - \sum_{m_\beta \in \beta(\bar{x})} \left( \bar{\sigma}_{1,m_\beta} F_{1,m_\beta}(x) + \bar{\sigma}_{2,m_\beta} F_{2,m_\beta}(x) \right) \qquad (5)$$

and $T_{\bar{x}}M(\bar{x})$ denotes the tangent space of $M(\bar{x})$ at $\bar{x}$,

$$\begin{aligned} T_{\bar{x}}M(\bar{x}) := \{\xi \in \mathbb{R}^n \mid \ & Dh_i(\bar{x})\,\xi = 0,\ i \in I, \\ & DF_{1,m_\alpha}(\bar{x})\,\xi = 0,\ m_\alpha \in \alpha(\bar{x}), \\ & DF_{2,m_\gamma}(\bar{x})\,\xi = 0,\ m_\gamma \in \gamma(\bar{x}), \\ & Dg_j(\bar{x})\,\xi = 0,\ j \in J_0(\bar{x}) \\ & DF_{1,m_\beta}(\bar{x})\,\xi = 0,\ DF_{2,m_\beta}(\bar{x})\,\xi = 0,\ m_\beta \in \beta(\bar{x})\}. \end{aligned}$$

Condition ND3 means that the matrix $V^T D^2 L(\bar{x}) V$ is nonsingular, where $V$ is some matrix whose columns form a basis for the tangent space $T_{\bar{x}}M(\bar{x})$.

**Definition 2.3 (C-index, cf. [12])** *Let $\bar{x} \in M$ be a nondegenerate C-stationary point with Lagrange multipliers as in Definition 2.2. The number of negative/positive eigenvalues of $D^2L(\bar{x}) \mid_{T_{\bar{x}}M(\bar{x})}$ is called the quadratic index (QI)/quadratic coindex (QCI) of $\bar{x}$. The number of pairs $(\bar{\sigma}_{1,m_\beta}, \bar{\sigma}_{2,m_\beta})$, $m_\beta \in \beta(\bar{x})$ with both $\bar{\sigma}_{1,m_\beta}$ and $\bar{\sigma}_{2,m_\beta}$ negative/positive is called the bi-active index (BI)/bi-active coindex (BCI) of $\bar{x}$. The number $(QI + BI)/(QCI + BCI)$ is called the Clarke-index (C-index)/Clarke-coindex (C-coindex) of $\bar{x}$.*

Note that in the absence of complementarity constraints, C-index has only the QI-part and coincides with the well-known quadratic index of a nondegenerate Karush-Kuhn-Tucker-point in nonlinear programming or, equivalently, with the Morse index (cf. [7, 8, 10]).

The following proposition uses the C-index for the characterization of a local minimizer. Its proof is omitted since it can be easily seen (see also [12, 13]).

**Proposition 2.1** *(i) Assume that $\bar{x}$ is a local minimizer for MPCC and that LICQ holds at $\bar{x}$. Then, $\bar{x}$ is a C-stationary point for MPCC.*

*(ii) Let $\bar{x}$ be a nondegenerate C-stationary point for MPCC. Then, $\bar{x}$ is a local minimizer for MPCC if and only if its C-index is equal to zero.*

The next proposition concerning genericity results for LICQ and for nondegenericity of C-stationary points mainly follows from [7]. It was shown in [14] and for the special case of QPCC in [12]. For its formulation we need some further definitions. The space $C^2(\mathbb{R}^n, \mathbb{R})$ will be topologized by means of the strong (or Whitney-) $C^2$-topology, denoted by $C^2_s$ (cf. [5, 7]). The $C^2_s$-topology is generated by allowing perturbations of the functions and their derivatives up to second order which are controlled by means of continuous positive functions $\varepsilon(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$. The product space $C^2(R^n, \mathbb{R}^l) \cong C^2(R^n, \mathbb{R}) \times \cdots \times C^2(R^n, \mathbb{R})$ will be topologized with the corresponding product topology.

**Proposition 2.2 (cf. [12, 14])**

*(i) Let $\mathcal{F}$ denote the subset of $C^2(\mathbb{R}^n, \mathbb{R}^{|I|}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|J|}) \times C^2(\mathbb{R}^n, \mathbb{R}^k) \times C^2(\mathbb{R}^n, \mathbb{R}^k)$ consisting of those $(h, g, F_1, F_2)$ for which LICQ holds at all points $x \in M[h, g, F_1, F_2]$. Then, $\mathcal{F}$ is $C^2_s$-open and -dense.*

*(ii) Let $\mathcal{D}$ denote the subset of $C^2(\mathbb{R}^n, \mathbb{R}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|I|}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|J|}) \times C^2(\mathbb{R}^n, \mathbb{R}^k) \times C^2(\mathbb{R}^n, \mathbb{R}^k)$ consisting of those $(f, h, g, F_1, F_2)$ for which each C-stationary point of MPCC with data functions $(f, h, g, F_1, F_2)$ is nondegenerate. Then, $\mathcal{D}$ is $C^2_s$-open and -dense.*

**Definition 2.4** *The feasible set $M$ admits a local $C^r$-coordinate system of $\mathbb{R}^n$ $(r \geq 1)$ at $\bar{x}$ by means of a $C^r$-diffeomorphism $\Phi : U \longrightarrow V$ with open $\mathbb{R}^n$-neighborhoods $U$ and $V$ of $\bar{x}$ and $0$, respectively, if it holds:*

*(i)* $\Phi(\bar{x}) = 0,$

*(ii)* $\Phi(M \cap U) = \left(\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial \mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p\right) \cap V.$

**Lemma 2.1 (cf. also [14])** *Suppose that LICQ holds at $\bar{x} \in M$. Then $M$ admits a local $C^2$-coordinate system of $\mathbb{R}^n$ at $\bar{x}$.*

**Proof**. Choose vectors $\xi_l \in \mathbb{R}^n$, $l = 1, \dots, p$, which form - together with the vectors

$$\{D^T h_i(\bar{x}),\, i \in I,,\, D^T F_{1,m_\alpha}(\bar{x}),\, m_\alpha \in \alpha(\bar{x}),\, D^T F_{2,m_\gamma}(\bar{x}),\, m_\gamma \in \gamma(\bar{x}),$$
$$D^T g_j(\bar{x}),\, j \in J_0(\bar{x}),\, D^T F_{1,m_\beta}(\bar{x}),\, D^T F_{2,m_\beta}(\bar{x}),\, m_\beta \in \beta(\bar{x})\}$$

- a basis for $\mathbb{R}^n$. Next we put

$$\left.\begin{array}{lcl} y_i & := & h_i(x),\, i \in I \\ y_{|I|+m_\alpha} & := & F_{1,m_\alpha}(x),\, m_\alpha \in \alpha(\bar{x}) \\ y_{|I|+m_\gamma} & := & F_{2,m_\gamma}(x),\, m_\gamma \in \gamma(\bar{x}) \\ y_{s+j} & := & g_j(x),\, j \in J_0(\bar{x}) \\ y_{s+|J_0(\bar{x})|+2m_\beta-1} & := & F_{1,m_\beta}(x) \\ y_{s+|J_0(\bar{x})|+2m_\beta} & := & F_{2,m_\beta}(x),\, m_\beta = 1, \dots, |\beta(\bar{x})| \\ y_{n-p+l} & := & \xi_l^T(x - \bar{x}),\, l = 1, \dots, p. \end{array}\right\} \qquad (6)$$

or, shortly,

$$y = \Phi(x). \qquad (7)$$

Note that $\Phi \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $\Phi(\bar{x}) = 0$ and the Jacobi-matrix $D\Phi(\bar{x})$ is nonsingular (in virtue of LICQ and the choice of $\xi_l$, $l = 1, \dots, p$). By means of the Implicit Function Theorem there exist open neighborhoods $U$ of $\bar{x}$ and $V$ of 0 such that $\Phi : U \longrightarrow V$ is a $C^2$-diffeomorphism. By shrinking $U$, if necessary, we can guarantee that $J_0(x) \subset J_0(\bar{x})$ and $\beta(x) \subset \beta(\bar{x})$ for all $x \in M \cap U$. Thus, the property (ii) in Definition 2.4 follows directly from the definition of $\Phi$. $\square$

**Definition 2.5** *We will refer to the $C^2$-diffeomorphism $\Phi$ defined by (6), (7) as standard-diffeomorphism.*

**Remark 2.1** *From the proof of Lemma 2.1 it follows that the Lagrange multipliers at a nondegenerate C-stationary point are the corresponding partial derivatives of the objective function in new coordinates given by the standard-diffeomorphism (cf. [6], Lemma 2.2.1). Moreover, the Hessian with respect to the last $p$ coordinates corresponds to the restriction of the Lagrange function's Hessian on the respective tangent space (cf. [6], Lemma 2.2.10).*

# 3 Main results

**Theorem 3.1 (Morse Lemma for MPCC)** *Suppose that $\bar{x}$ is a nondegenerate C-stationary point for MPCC with quadratic index QI, bi-active index BI and C-index = QI + BI. Then, there exists a local $C^1$-coordinate system $\Psi : U \longrightarrow V$ of $\mathbb{R}^n$ around $\bar{x}$ (according to Definition 2.4) such that:*

$$f \circ \Psi^{-1}(0_s, y_{s+1}, \ldots, y_n) = f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_{i+s} + \sum_{j=1}^{|\beta(\bar{x})|} \pm(y_{2j+q-1} + y_{2j+q}) + \sum_{k=1}^{p} \pm y^2_{k+n-p}, \tag{8}$$

*where $y \in \{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$. Moreover, in (8) there are exactly BI negative linear pairs and QI negative squares.*

**Proof**. W.l.o.g., we may assume $f(\bar{x}) = 0$. Let $\Phi : U \longrightarrow V$ be a standard-diffeomorphism according to Definition (2.5). We put $\bar{f} := f \circ \Phi^{-1}$ on the set $\left(\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p\right) \cap V$. From now on we may assume $s = 0$. In view of Remark 2.1 we have at the origin:

(i) $\dfrac{\partial \bar{f}}{\partial y_i} > 0$, $i \in J_0(\bar{x})$,

(ii) $\dfrac{\partial \bar{f}}{\partial y_{2j+q-1}} \cdot \dfrac{\partial \bar{f}}{\partial y_{2j+q}} > 0$, $j = 1, \ldots |\beta(\bar{x})|$,

(iii) $\dfrac{\partial \bar{f}}{\partial y_{2j+q-1}} < 0$ for exactly BI indices $j \in \{1, \ldots |\beta(\bar{x})|\}$,

(iv) $\dfrac{\partial \bar{f}}{\partial y_{k+n-p}} = 0$, $k = 1, \ldots, p$ and $\left(\dfrac{\partial^2 \bar{f}}{\partial y_{k_1+n-p} \partial y_{k_2+n-p}}\right)_{1 \le k_1, k_2 \le p}$ is a nonsingular matrix with QI negative eigenvalues.

From now on we denote $\bar{f}$ by $f$. Under the following coordinate transformations the set $\mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ will be transformed in itself (equivariance). As an abbreviation we put $y = (Y_{n-p}, Y^p)$, where $Y_{n-p} = (y_1, \ldots, y_{n-p})$

and $Y^p = (y_{n-p+1}, \dots, y_n)$. We write

$$f(Y_{n-p}, Y^p) = f(0, Y^p) + \int_0^1 \frac{d}{dt} f(tY_{n-p}, Y^p)dt = f(0, Y^p) + \sum_{i=1}^{n-p} y_i g_i(y),$$

where $g_i \in C^1$, $i = 1, \dots, n-p$.

In view of (iv) we may apply the Morse Lemma on the $C^2$-function $f(0, Y^p)$ (cf. [7], Theorem 2.8.2) without affecting the coordinates $Y_{n-p}$. The corresponding coordinate transformation is of class $C^1$. Denoting the transformed functions $f$, $g_j$ again by $f$, $g_j$, we obtain:

$$f(y) = \sum_{i=1}^{n-p} y_i g_i(y) + \sum_{k=1}^{p} \pm y_{k+n-p}^2.$$

Note that $g_i(0) = \dfrac{\partial f}{\partial y_i}(0)$, $i = 1, \dots, n-p$. Recalling (i)-(iii, we have

$$y_i |g_i(y)|, \; i = 1, \dots, n-p, \quad y_j, \; j = n-p+1, \dots, n \tag{9}$$

as new local $C^1$-coordinates. Denoting the transformed function $f$ again by $f$ and, recalling the signs in (i)-(iii), we obtain (8). Here, the coordinate transformation $\Psi$ is understood as the composition of all previous ones. □

Theorem 3.1 allows us to provide two other local representations (normal forms) of the objective function on the MPCC feasible set with respect to Lipschitz and Hölder coordinate systems.

Recall that the set $\partial \mathbb{H}^2$ represents the complementarity relations

$$u \geq 0, \; v \geq 0, \; u \cdot v = 0.$$

Define the mapping $\varphi : \partial \mathbb{H}^2 \longrightarrow \mathbb{R}^1 \times 0_1$ as follows:

$$\varphi(u, 0) := (u, 0), \; \varphi(0, v) := (-v, 0). \tag{10}$$

Coordinatewise extension of $\varphi$ on $(\partial \mathbb{H}^2)^{|\beta(\bar{x})|}$ and leaving the other coordinates invariant, (10) induces the Lipschitz coordinate transformation $\Phi$,

$$\Phi : \{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times \left(\partial \mathbb{H}^2\right)^{|\beta(\bar{x})|} \times \mathbb{R}^p \longrightarrow \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p \tag{11}$$

In the right-hand side of (11) the zeros $\{0_s\}$ and $\{0_1\}$ ($|\beta(\bar{x})|$-times) are deleted. The proof of the following corollary is now straightforward.

**Corollary 3.1** *Let $f$ have the normal form as in (8) and let $\Phi$ be the Lipschitz coordinate transformation (11). Then, we have:*

$$f \circ \Phi^{-1}(y) = f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_i + \sum_{j=|J_0(\bar{x})|+1}^{J_0(\bar{x})+|\beta(\bar{x})|} \pm |y_j| + \sum_{k=J_0(\bar{x})+|\beta(\bar{x})|+1}^{n-|\beta(\bar{x})|+s} \pm y^2_{k+n-p}. \quad (12)$$

*In (12) there are exactly BI negative absolute value terms and QI negative squares.*

On $\mathbb{R}^1$ we introduce the transformation $\psi$:

$$\psi(y) := sgn(y)\sqrt{|y|}. \quad (13)$$

Note that the function $\pm|y|$ transforms into $\pm y^2$. Coordinatewise extension of $\psi$ on $\mathbb{R}^{|\beta(\bar{x})|}$ and leaving the other coordinates invariant, (13) induces the Hölder coordinate transformation $\Psi$,

$$\Psi : \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p \longrightarrow \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p \quad (14)$$

The proof of the following corollary is again straightforward.

**Corollary 3.2** *Let $f$ have the normal form as in (12) and let $\Psi$ be the Hölder coordinate transformation (14). Then, we have:*

$$f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_i + \sum_{j=|J_0(\bar{x})|+1}^{n-|\beta(\bar{x})|+s} \pm y^2_j. \quad (15)$$

*The number of negative squares in (15) equals the C-index BI+QI.*

Now, we come to the main theorem of the paper. For the topological concepts we refer to [7, 15]. For $a, b \in \mathbb{R}$, $a < b$ define the sets

$$M^a := \{x \in M \mid f(x) \le a\}$$

and

$$M^b_a := \{x \in M \mid a \le f(x) \le b\}.$$

**Theorem 3.2** *Let $M_a^b$ be compact and suppose that LICQ is satisfied at all points $x \in M_a^b$.*

*(a)* **(Deformation Theorem)** *If $M_a^b$ does not contain any C-stationary point for MPCC, then $M^a$ is a strong deformation retract of $M^b$.*

*(b)* **(Cell-attachment Theorem)** *If $M_a^b$ contains exactly one C-stationary point for MPCC, say $\bar{x}$, and if $a < f(\bar{x}) < b$ and the C-index of $\bar{x}$ is equal to $q$, then $M^b$ is homotopy-equivalent to $M^a$ with a $q$-cell attached.*

**Proof.** (a) Due to LICQ at all $x \in M_a^b$ there exist real numbers

$\lambda_i(x)$, $i \in I$, $\varrho_{m_\alpha}(x)$, $m_\alpha \in \alpha(x)$, $\vartheta_{m_\gamma}(x)$, $m_\gamma \in \gamma(x)$, $\mu_j(x)$, $j \in J_0(x)$,

$\sigma_{1,m_\beta}(x)$, $\sigma_{2,m_\beta}(x)$, $m_\beta \in \beta(x)$, $\nu_l(x)$, $l = 1, \ldots, p$ such that:

$$Df(x) = \sum_{i\in I}\lambda_i(x)Dh_i(x) + \sum_{m_\alpha\in\alpha(x)}\varrho_{m_\alpha}(x)DF_{1,m_\alpha}(x) + \sum_{m_\gamma\in\gamma(x)}\vartheta_{m_\gamma}(x)DF_{2,m_\gamma}(x)$$

$$+\sum_{j\in J_0(x)}\mu_j(x)Dg_j(x) + \sum_{m_\beta\in\beta(x)}\big(\sigma_{1,m_\beta}(x)DF_{1,m_\beta}(x) + \sigma_{2,m_\beta}(x)DF_{2,m_\beta}(x)\big) + \sum_{l=1}^{p}\nu_l(x)\,\xi_l,$$

where vectors $\xi_l$, $l = 1, \ldots, p$ are chosen as in Lemma 2.1. We set:

$A := \{x \in M_a^b \mid \text{ there exists } l \in \{1, \ldots, p\} \text{ with } \nu_l(x) \neq 0\}$,

$B := \{x \in M_a^b \mid \text{ there exists } j \in J_0(x) \text{ with } \mu_j(x) < 0\}$,

$C := \{x \in M_a^b \mid \text{ there exists } m_\beta \in \beta(x) \text{ with } \sigma_{1,m_\beta}(x) \cdot \sigma_{2,m_\beta}(x) < 0\}$.

Since each $\bar{x} \in M_a^b$ is not C-stationary for MPCC, we get $\bar{x} \in A \cup B \cup C$.

The proof consists of a local argument and its globalization.

First, we show the **local argument:**

For each $\bar{x} \in M_a^b$ there exist an $(R^n)$-neighborhood $U_{\bar{x}}$ of $\bar{x}$, $t_{\bar{x}} > 0$ and a mapping

$$\Psi^{\bar{x}} : \left\{ \begin{array}{ccc} [0, t_{\bar{x}}) \times M^b \cap U_{\bar{x}} & \longrightarrow & M \\ (t, x) & \mapsto & \Psi^{\bar{x}}(t, x) \end{array} \right. \quad \text{such that:}$$

(i) $\Psi^{\bar{x}}(t, M^b \cap U_{\bar{x}}) \subset M^{b-t}$ for all $t \in [0, t_{\bar{x}})$,

(ii) $\Psi^{\bar{x}}(t_1+t_2, \cdot) = \Psi^{\bar{x}}(t_1, \Psi^{\bar{x}}(t_2, \cdot))$ for all $t_1,\ t_2 \in [0, t_{\bar{x}})$ with $t_1+t_2 \in [0, t_{\bar{x}})$,

(iii) if $\bar{x} \in A \cup B$, then $\Psi^{\bar{x}}(\cdot, \cdot)$ is a $C^1$-flow corresponding to a $C^1$-vector field $F^{\bar{x}}$.

(iv) if $\bar{x} \in C$, then $\Psi^{\bar{x}}(\cdot, \cdot)$ is a Lipschitz flow.

Obviously, the level sets of $f$ are locally mapped onto the level sets of $f \circ \Phi^{-1}$, where $\Phi$ is a $C^1$-diffeomorphism according to Definition 2.4. Applying the standard-diffeomorphism $\Phi$ from Definition 2.5, we consider $f \circ \Phi^{-1}$ (denoted by $f$ again). Thus, we have $\bar{x} = 0$ and $f$ is given on the feasible set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial \mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$.

**Case a)** $\bar{x} \in A$

Then, due to Remark 2.1 there exists $l \in \{1, \ldots, p\}$ with $\frac{\partial f}{\partial x_l}(\bar{x}) \neq 0$. Define a local $C^1$-vector field $F^{\bar{x}}$ as follows:

$$F^{\bar{x}}(x_1, \ldots, x_l, \ldots, x_n) := \left(0, \ldots, -\frac{\partial f}{\partial x_l}(x) \cdot \left(\frac{\partial f}{\partial x_l}(x)\right)^{-2}, \ldots, 0\right)^T.$$

After respective inverse changes of local coordinates $F^{\bar{x}}$ induces the flow $\Psi^{\bar{x}}$, which fits the local argument (see [7], Theorem 2.7.6 for details).

**Case b)** $\bar{x} \in B$

Then, due to Remark 2.1 there exists $j \in J_0(x)$ with $\frac{\partial f}{\partial x_j}(\bar{x}) < 0$. By means of a $C^1$-coordinate transformation (along the lines of [7], Theorem 3.2.26) in the $j$-th coordinate on $\mathbb{H}$, leaving the other coordinates unchanged, we obtain locally for $f$:

$$f(x_1, \ldots, x_j, \ldots, x_n) = -x_j + f(x_1, \ldots, \bar{x}_j, \ldots, x_n).$$

Define a local $C^1$-vector field $F^{\bar{x}}$ as follows:

$$F^{\bar{x}}(x_1, \ldots, x_j, \ldots, x_n) := (0, \ldots, 1, \ldots, 0)^T.$$

After respective inverse changes of local coordinates $F^{\bar{x}}$ induces the flow $\Psi^{\bar{x}}$, which fits the local argument (see [7], Theorem 3.3.25 for details).

**Case c)** $\bar{x} \in C$

Then, due to Remark 2.1 there exists $m_\beta \in \beta(x)$ with

$$\frac{\partial f}{\partial x_{1,m_\beta}}(\bar{x}) \cdot \frac{\partial f}{\partial x_{1,m_\beta}}(\bar{x}) < 0.$$

W.l.o.g., we assume that $\frac{\partial f}{\partial x_{1,m_\beta}}(\bar{x}) < 0$ and $\frac{\partial f}{\partial x_{1,m_\beta}}(\bar{x}) > 0$.

From the proof of Theorem 3.1, Formula (9) we can obtain for $f$ in new $C^1$-coordinates the representation:

$$f(x_1, \ldots, x_j, \ldots, x_n) = -x_{1,m_\beta} + x_{2,m_\beta} + f(x_1, \ldots, \bar{x}_{1,m_\beta}, \bar{x}_{2,m_\beta}, \ldots, x_n).$$

Define the mapping $\Psi^{\bar{x}}$ locally as follows:

$$\Psi^{\bar{x}}(t, x_1, \ldots, x_{1,m_\beta}, x_{2,m_\beta}, \ldots, x_n) :=$$

$$(x_1, \ldots, x_{1,m_\beta} + \max\{0, t - x_{2,m_\beta}\}, \max\{0, x_{2,m_\beta} - t\}, \ldots, x_n)^T.$$

After respective inverse changes of local coordinates $\Psi^{\bar{x}}$ fits the local argument.

Note that in all the Cases a)-c) $\Psi^{\bar{x}}(t, \cdot)$ leaves the feasible set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ invariant.

**Globalization.**

Consider the open covering $\{U_x \,|\, x \in C\} \cup \{U_{\bar{x}} \,|\, \bar{x} \in M_a^b \backslash \{U_x \,|\, x \in C\}\}$ of $M_a^b$. Due to continuity arguments $U_{\bar{x}}$, $\bar{x} \in M_a^b \backslash \{U_x \,|\, x \in C\}$ can be taken smaller, if necessary, to be disjoint with $C$. Since $M_a^b$ is compact, we get a finite open subcovering $\{U_{x_i} \,|\, x_i \in C\} \cup \{U_{\bar{x}_j} \,|\, \bar{x}_j \in M_a^b \backslash \{U_x \,|\, x \in C\}\}$ of $M_a^b$. Using a $C^\infty$-partition of unity $\{\phi_j\}$ subordinate to $\{U_{\bar{x}_j} \,|\, \bar{x}_j \in M_a^b \backslash \{U_x \,|\, x \in C\}\}$ we define with $F^{\bar{x}_j}$ (cf. Cases a),b)) a $C^1$-vector field $F := \sum_j \phi_j F^{\bar{x}_j}$. The last induces a flow $\Psi$ on $\{U_{\bar{x}_j} \,|\, \bar{x}_j \in M_a^b \backslash \{U_x \,|\, x \in C\}\}$ (see [7], Theorem 3.3.14 for details). Note that in each nonempty overlapping region $U_{x_i} \cap U_{x_j}$, $x_i \in C$,

$x_j \in M_a^b \backslash \{U_x \,|\, x \in C\}$ the flow $\Psi^{x_i}$ induces exactly the vector field $F$ (cf. Case c)). Hence, local trajectories can be glued together on $M_a^b$, named by $\Psi$ again. Moreover, moving along the local pieces of the trajectories $\Psi(\cdot, x)$, $x \in M_a^b$ reduces the level of $f$ at least by a positive real

$$\frac{\min\{t_{x_i}, t_{x_j} \,|\, x_i \in C, x_j \in M_a^b \backslash \{U_x \,|\, x \in C\}\}}{2}.$$

Thus, we obtain for $x \in M_a^b$ a unique $t_a(x) > 0$ with $\Psi(t_a(x), x) \in M^a$. It is not hard (but technical) to realize that $t_a : x \longrightarrow t_a(x)$ is Lipschitz. Finally, we define $r : [0,1] \times M^b \longrightarrow M^b$ as follows:

$$r(\tau, x) : \left\{ \begin{array}{lll} x & \text{for } x \in M^a, & \tau \in [0,1] \\ \Psi(\tau t_a(x), x) & \text{for } x \in M_a^b, & \tau \in [0,1]. \end{array} \right.$$

The mapping $r$ provides that $M^a$ is a strong deformation retract of $M^b$.

(b) In virtue of the Deformation Theorem and the normal forms (8), (12), (15), the proof of the Cell-attachment part becomes standard. In fact, the Deformation Theorem allows deformations up to an arbitrarily small neighborhood of the C-stationary point $\bar{x}$. In such a neighborhood we can work in continuous local coordinates, and use the explicit normal form (15). In the normal form (15) the origin is a non-degenerate KKT-point and the cell-attachment can be performed as in [7], Theorem 3.3.33. $\Box$

**Remark 3.2** *We emphasize that the linear terms $y_i$, $i \in J_0(\bar{x})$, in (15) do not contribute to the dimension of the cell to be attached. In fact, w.r.t. lower level sets, the* 1*-dim. constrained singularity $y$, $y \geq 0$, plays the same role as the unconstrained singularity $y^2$. In this sense the constrained linear terms in (15) do not contribute to the number of negative squares.*

**Remark 3.3** *Another way of looking at the cell-attachment part is via stratified Morse Theory ([4], Section 3.7). In fact, recall the normal form (8). The set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial \mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ can be interpreted as the product of the "tangential part" $\mathbb{R}^p$ and the "normal part" $\mathbb{H}^{|J_0(\bar{x})|} \times (\partial \mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$. The main theorem in [4] states that the local "Morse data" is the product of the tangential "Morse data" with the normal "Morse data". The tangential Morse index equals QI and, in view of Remark 3.2, the normal Morse index equals BI. In the product, the index then becomes the sum QI+BI, what is precisely the C-index.*

# 4 Discussion of different stationarity concepts

We briefly review well-known definitions of various stationarity concepts and connections between them (cf. [2], [11], [13]).

**Definition 4.1** *Let* $\bar{x} \in M$*.*

*(i)* $\bar{x}$ *is called W-stationary if (2), (3) hold.*

*(ii)* $\bar{x}$ *is called A-stationary if (2), (3) hold and*

$$\bar{\sigma}_{1,m_\beta} \geq 0 \text{ or } \bar{\sigma}_{2,m_\beta} \geq 0 \quad \text{for all } m_\beta \in \beta(\bar{x}).$$

*(iii)* $\bar{x}$ *is called M-stationary if (2), (3) hold and*

$$(\bar{\sigma}_{1,m_\beta} > 0 \text{ and } \bar{\sigma}_{1,m_\beta} > 0) \text{ or } \bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} = 0 \quad \text{for all } m_\beta \in \beta(\bar{x}).$$

*(iv)* $\bar{x}$ *is called S-stationary if (2), (3) hold and*

$$\bar{\sigma}_{1,m_\beta} \geq 0, \bar{\sigma}_{2,m_\beta} \geq 0 \quad \text{for all } m_\beta \in \beta(\bar{x}).$$

*(v)* $\bar{x}$ *is called B-stationary if* $d = 0$ *is a local solution of the linearized problem:*

$$\min\ f(\bar{x}) + Df(\bar{x})d \quad \text{s.t.}$$

$$\begin{cases} F_{1,m}(\bar{x}) + DF_{1,m}(\bar{x})d \geq 0, F_{2,m}(\bar{x}) + DF_{2,m}(\bar{x})d \geq 0, \\ (F_{1,m}(\bar{x}) + DF_{1,m}(\bar{x})d) \cdot (F_{2,m}(\bar{x}) + DF_{2,m}(\bar{x})d) = 0,\ m = 1, \ldots, k, \\ h(\bar{x}) + Dh(\bar{x})d = 0, g(\bar{x}) + Dg(\bar{x})d \geq 0. \end{cases}$$

The following diagram summarizes the relations between mentioned stationarity concepts (e.g. [16]):

$$\begin{tikzcd}
 & \textbf{S-stationary point} \arrow[rr, Leftrightarrow, "\text{under LICQ}"'] \arrow[d, Rightarrow] & & \textbf{B-stationary point} \\
 & \textbf{M-stationary point} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
\textbf{C-stationary point} \arrow[dr, Rightarrow] & & \textbf{A-stationary point} \arrow[dl, Rightarrow] & \\
 & \textbf{W-stationary point} & &
\end{tikzcd}$$

Assuming nondegeneracy (as in Definition 2.2) we see that A-, M, S-, B-stationary points are tighter characterizations of a local minimum than C-stationary points. However, they exclude C-stationary points with $BI > 0$. These points are also crucial for the topological structure of MPCC (cf. Cell-attachment Theorem). For global optimization, points of $C\text{-}index = 1$ play an important role, see also the Introduction. We emphasize that among the points of $C\text{-}index = 1$ there is no substantial difference between the points with $BI = 1$, $QI = 0$ and $BI = 0$, $QI = 1$. It is worth to mention that a linear descent direction might exist in a nondegenerate C-stationary point (see [9] and [13] for examples and the following discussion). However, at points with $BI = 1$, $QI = 0$ there are exactly two directions of linear decrease. Both of them are important from a global point of view. In turn, W-stationary points contain those with negative and positive Lagrange multipliers corresponding to the same complementarity constraint. Due to Deformation Theorem such points are irrelevant for the topological structure of MPCC.

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