

# Constrained School Choice\*

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## Abstract

Recently, several school districts in the US have adopted or consider adopting the Student-Optimal Stable mechanism or the Top Trading Cycles mechanism to assign children to public schools. There is evidence that for school districts that employ (variants of) the so-called Boston mechanism the transition would lead to efficiency gains. The first two mechanisms are strategy-proof, but in practice student assignment procedures typically impede a student to submit a preference list that contains all his acceptable schools. We study the preference revelation game where students can only declare up to a fixed number of schools to be acceptable. We focus on the stability of the Nash equilibrium outcomes. Our main results identify rather stringent necessary and sufficient conditions on the priorities to guarantee stability. This stands in sharp contrast with the Boston mechanism which yields stable Nash equilibrium outcomes, independently of the quota. Hence, the transition to any of the two mechanisms is likely to come with a higher risk that students seek legal action as lower priority students may occupy more preferred schools.

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# 1 Introduction

School choice is referred in the literature on education as giving parents a say in the choice of the schools their children will attend. A recent paper by Abdulkadiroğlu and Sönmez (2003) has led to an upsurge of enthusiasm in the use of matching theory for the design and study of school choice mechanisms.<sup>1</sup> Abdulkadiroğlu and Sönmez (2003) discuss critical flaws of the current procedures of some school districts in the US to assign children to public schools, pointing out that the widely used Boston mechanism has the serious shortcoming that it is not in the parents' best interest to reveal their true preferences. Using a mechanism design approach, they propose and analyze two alternative student assignment mechanisms that do not have this shortcoming: the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism.

Since real-life school choice situations typically involve a large number of participants and a relatively small number of school programs, parents are often asked to submit a preference list containing only a limited number of schools.<sup>2</sup> This restriction is reason for concern, for complete revelation of one's true preferences is typically no longer an option in this case. In other words, the argument that the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism are strategy-proof is no longer valid. Imposing a curb on the length of the submitted lists, though certainly having the merit of "simplifying" matters, has the perverse effect of forcing participants to not be truthful, and eventually compel them to adopt a strategic behavior when choosing which ordered list to submit. We are then back in the situation of the Boston mechanism where participants are forced to play a complicated admissions game. Participants may adopt strategic behavior because the "quantitative" effect (*i.e.*, participants cannot reveal their complete preference lists) is likely to have a "qualitative" effect (*i.e.*, participants may self-select by not declaring their most preferred options). For instance, if a participant fears rejection by his most preferred programs, it can be advantageous not to apply to these programs and use instead its allowed application slots for less preferred programs.

The goal of this paper is to explore the effects of imposing a *quota* (*i.e.*, a maximal length) on the submittable preference lists of students. Thereby we revive an issue that

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<sup>1</sup>Recent papers include Abdulkadiroğlu (2005), Abdulkadiroğlu, Pathak, and Roth (2005), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005), Chen and Sönmez (2006), Erdil and Ergin (2006), Ergin and Sönmez (2006), Kesten (2005), and Pathak and Sönmez (2006).

<sup>2</sup>For instance, in the school district of New York City each year more than 90,000 students are assigned to about 500 school programs, and parents are asked to submit a preference list containing at most 12 school programs.

was initially discussed by Romero-Medina (1998).<sup>3</sup> To this end, we study school choice problems (Abdulkadiroğlu and Sönmez, 2003) where a number of students has to be assigned to a number of schools, each of which has a limited seat capacity. Students have preferences over schools and remaining unassigned and schools have exogenously given priority rankings over students.<sup>4</sup> We introduce a preference revelation game where students can only declare up to a fixed number (the quota) of schools to be acceptable. Each possible quota, from 1 up to the total number of schools, together with a student assignment mechanism induces a strategic “quota-game.” Since the presence of the quota eliminates the existence of a dominant strategy when the mechanism at hand is the Student-Optimal Stable or Top Trading Cycles mechanism, we focus our analysis on the Nash equilibria of the quota-games. Regarding the Student-Optimal Stable mechanism, our approach nicely complements the work of Roth (1984a), Gale and Sotomayor (1985a), and Alcalde (1996) who characterized the set of Nash equilibrium outcomes when the schools are strategic.<sup>5</sup>

For all three mechanisms and for any quota, Nash equilibria in pure strategies exist. For the Boston mechanism this follows from a straightforward extension of a result due to Ergin and Sönmez (2006), whereas for Student-Optimal Stable mechanism the existence of Nash equilibria in pure strategies was already established by Romero-Medina (1998). We also establish that the associated quota-games have a common feature: the equilibria

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<sup>3</sup>To the best of our knowledge, Romero-Medina (1998) is the only paper that explicitly analyzes restrictions on the length of submittable preference lists in the same setting. He focuses exclusively on the Student-Optimal Stable mechanism and establishes that the correspondence of stable matchings is implemented in Nash equilibria, independently of the quota (Romero-Medina, 1998, Theorem 7). While we show that any stable matching can be sustained at some Nash equilibrium (Proposition 6.2), Nash equilibria can also yield unstable matchings (Example 6.3). A recent paper by Kojima and Pathak (2006) takes a statistical approach to study manipulations by *schools* in the Student-Optimal Stable mechanism with restrictions on the length of submittable preference lists of schools.

<sup>4</sup>Very often local or state laws determine the priority rankings. Typically, students who live closer to a school or have siblings attending a school have higher priority to be admitted at the school. In other situations, priority rankings may be determined by one or several entrance exams. Then students who achieve higher test scores in the entrance exam of a school have higher priority for admission at the school than students with lower test scores.

<sup>5</sup>Roth (1984a) and Gale and Sotomayor (1985a) characterized the set of Nash equilibrium outcomes when schools are strategic agents in a college admission problem, assuming that students truthfully reveal their (whole) preferences. In particular, they showed that Nash equilibria yield stable matchings and that any stable matching can be obtained as a Nash equilibrium outcome. Alcalde (1996) went one step further assuming that students may not necessarily use their weakly dominant strategy. He showed that the set of Nash equilibrium outcomes coincides with the set of individually rational matchings.

are nested with respect to the quota. More precisely, given a quota, any Nash equilibrium is also a Nash equilibrium under any less stringent quota. This leads to the following important observation: If a Nash equilibrium outcome in a quota-game has an undesirable property then this is not simply due to the presence of a constraint on the size of submittable lists. Regarding the Boston and the Top Trading Cycle mechanism, we obtain a much stronger result: Nash equilibrium outcomes are independent of the quota. This is a powerful result, since it allows us to reduce the analysis to games of quota 1. Besides, existence of a Nash equilibrium for the Top Trading Cycles mechanism easily follows from this result and the strategy-proofness in the unconstrained setting.

Stability is the central concept in the two-sided matching literature and does not lose its importance in the closely related model of school choice.<sup>6</sup> Loosely speaking, stability of an assignment obtains when, for any student, all the schools he prefers to the one he is assigned to have exhausted their capacity with students that have higher priority. Hence, if an assignment is not stable then a student might seek legal action against the school district authorities for not getting assigned a seat which is either unfilled or filled by a student with a lower priority. Regarding the Boston mechanism, it is easy to show that the correspondence of stable matchings is implemented in Nash equilibria. The equilibrium analysis turns out to be quite different for the other two mechanisms: under both mechanisms there can be unstable Nash equilibrium outcomes. In fact, we exhibit a school choice problem with a (strong) Nash equilibrium in (undominated) truncation strategies<sup>7</sup> that yields an unstable matching. Yet, the two mechanisms are different in another aspect: unlike the Top Trading Cycles mechanism, under the Student-Optimal Stable mechanism any stable matching can be sustained at some Nash equilibrium, independently of the quota. We show that stability of Nash equilibrium outcomes is guaranteed if and only if schools' priorities satisfy Ergin's (2002) and Kesten's (2006) acyclicity conditions, for the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism, respectively. This is quite surprising in light of Ergin's (2002) result that his acyclicity condition is necessary and sufficient for the Pareto-efficiency of the Student-Optimal Stable mechanism, and Kesten's (2006) result that his acyclicity condition is necessary and sufficient for the stability of the Top Trading Cycles mechanism.

The remainder of the paper is organized as follows. In Section 2, we recall the model of

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<sup>6</sup>In many centralized labor markets, clearinghouses are most often successful if they produce stable matchings — see Roth (2002) and the references therein.

<sup>7</sup>A truncation strategy is a list obtained from the true preferences by omitting a “tail,” *i.e.*, leaving out one particular school and all less preferred schools. These strategies have been studied by Roth and Rothblum (1999) and Ehlers (2004).

school choice. In Section 3, we describe the three mechanisms and provide an illustrative example. In Section 4, we introduce the strategic game induced by the imposition of a quota on the revealed preferences. In Sections 5, 6, and 7, we present our results on the existence, nestedness, and stability of the Nash equilibrium outcomes for the quota-game under the Boston, Student-Optimal Stable, and Top Trading Cycles mechanism, respectively. In Section 8, we study Nash equilibria in undominated truncation strategies for the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism. Finally, in Section 9, we discuss the policy implications of our results and our contribution to the literature on school choice. All proofs are relegated to the Appendices.

## 2 School Choice

Following Abdulkadiroğlu and Sönmez (2003) we define a *school choice problem* by a set of schools and a set of students, each of which has to be assigned a seat at not more than one of the schools. Each student is assumed to have strict preferences over the schools and the option of remaining unassigned. Each school is endowed with a strict priority ordering over the students and a fixed capacity of seats. Formally, a *school choice problem* is a 5-tuple  $(I, S, q, P, f)$  that consists of

1. a set of *students*  $I = \{i_1, \dots, i_n\}$ ,
2. a set of *schools*  $S = \{s_1, \dots, s_m\}$ ,
3. a *capacity* vector  $q = (q_{s_1}, \dots, q_{s_m})$ ,
4. a profile of strict *student preferences*  $P = (P_{i_1}, \dots, P_{i_n})$ , and
5. a strict *priority structure* of the schools over the students  $f = (f_{s_1}, \dots, f_{s_m})$ .

We denote by  $i$  and  $s$  a generic student and a generic school, respectively. An agent is an element of  $V := I \cup S$ . A generic agent is denoted by  $v$ . With a slight abuse of notation we write  $v$  for singletons  $\{v\} \subseteq V$ .

The preference relation  $P_i$  of student  $i$  is a linear order over  $S \cup i$ , where  $i$  denotes his outside option (*e.g.*, going to a private school). Student  $i$  prefers school  $s$  to school  $s'$  if  $sP_i s'$ . School  $s$  is *acceptable* to  $i$  if  $sP_i i$ . Henceforth, when describing a particular preference relation of a student we will only represent acceptable schools. For instance,  $P_i = s, s'$  means that student  $i$ 's most preferred school is  $s$ , his second best  $s'$ , and any other school is unacceptable. For the sake of convenience, if all schools are unacceptable for  $i$  then we sometimes write  $P_i = i$  instead of  $P_i = \emptyset$ . Let  $R_i$  denote the weak preference relation associated with the preference relation  $P_i$ .

The *priority ordering*  $f_s$  of school  $s$  assigns ranks to students according to their priority for school  $s$ . The *rank* of student  $i$  for school  $s$  is  $f_s(i)$ . Then,  $f_s(i) < f_s(j)$  means that *student  $i$  has higher priority (or lower rank) for school  $s$  than student  $j$* . For  $s \in S$  and  $i \in I$ , we denote by  $U_s^f(i)$  the set of students that have higher priority than student  $i$  for school  $s$ , *i.e.*,  $U_s^f(i) = \{j \in I : f_s(j) < f_s(i)\}$ .

Throughout the paper we fix the set of students  $I$  and the set of schools  $S$ . Hence, a *school choice problem* is given by a triple  $(P, f, q)$ , and simply by  $P$  when no confusion is possible.

School choice is closely related to the college admissions model (Gale and Shapley, 1962). The only but key difference between the two models is that in school choice schools are mere “objects” to be consumed by students, whereas in the college admissions model (or more generally, in two-sided matching) both sides of the market are agents with preferences over the other side. In other words, a college admissions problem is given by 1–4 above and 5’ below:

5’. a profile of strict school preferences  $P_S = (P_{s_1}, \dots, P_{s_m})$ ,

where  $P_s$  denotes the strict preference relation of school  $s \in S$  over the students.

Priority orderings in school choice can be reinterpreted as school preferences in the college admissions model. Therefore, many results or concepts for the college admissions model have their natural counterpart for school choice.<sup>8</sup> In particular, an outcome of a school choice or college admissions problem is a *matching*  $\mu : I \cup S \rightarrow 2^I \cup S$  such that for any  $i \in I$  and any  $s \in S$ ,

- $\mu(i) \in S \cup i$ ,
- $\mu(s) \in 2^I$ ,
- $\mu(i) = s$  if and only if  $i \in \mu(s)$ , and
- $|\mu(s)| \leq q_s$ .

For  $v \in V$ , we call  $\mu(v)$  agent  $v$ ’s allotment. For  $i \in I$ , if  $\mu(i) = s \in S$  then student  $i$  is *assigned* a seat at school  $s$  under  $\mu$ . If  $\mu(i) = i$  then student  $i$  is *unassigned* under  $\mu$ .<sup>9</sup> For convenience we often write a matching as a collection of sets. For instance,  $\mu = \{\{i_1, i_2, s_1\}, \{i_3\}, \{i_4, s_2\}\}$  denotes the matching in which students  $i_1$  and  $i_2$  each are assigned a seat at school  $s_1$ , student  $i_3$  is unassigned, and student  $i_4$  is assigned a seat at school  $s_2$ .

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<sup>8</sup>See, for instance, Balinski and Sönmez (1999).

<sup>9</sup>Education at the primary level, affirmed as a human right in the 1948 Universal Declaration of Human Rights, is compulsory in most countries. One of our results is that in equilibrium no student is unassigned if there is a vacant seat at some acceptable school.

A key property of matchings in the two-sided matching literature is stability. Informally, a matching is stable if, for any student, all the schools he prefers to the one he is assigned to have exhausted their capacity with students that have higher priority. Formally, let  $P$  be a school choice problem. A matching  $\mu$  is stable if

- it is *individually rational*, *i.e.*, for all  $i \in I$ ,  $\mu(i)R_i i$ ,
- it is *non wasteful* (Balinski and Sönmez, 1999), *i.e.*, for all  $i \in I$  and all  $s \in S$ ,  $sP_i\mu(i)$  implies  $|\mu(s)| = q_s$ , and
- there is no *justified envy*, *i.e.*, for all  $i, j \in I$  with  $\mu(j) = s \in S$ ,  $sP_i\mu(i)$  implies  $f_s(j) < f_s(i)$ .

We denote the set of individually rational matchings by  $IR(P)$ , the set of non wasteful matchings by  $NW(P)$ , and the set of stable matchings by  $S(P)$ .

Another desirable property for a matching is Pareto-efficiency. In the context of school choice, the schools are mere “objects.” Therefore, to determine whether a matching is Pareto-efficient we only take into account students’ welfare. A matching  $\mu'$  Pareto dominates a matching  $\mu$  if all students prefer  $\mu'$  to  $\mu$  and there is at least one student that strictly prefers  $\mu'$  to  $\mu$ . Formally,  $\mu'$  *Pareto dominates*  $\mu$  if  $\mu'(i)R_i\mu(i)$  for all  $i \in I$ , and  $\mu'(i')P_i'\mu(i')$  for some  $i' \in I$ . A matching is *Pareto-efficient* if it is not Pareto dominated by any other matching.

A (student assignment) *mechanism* systematically selects a matching for each school choice problem. A mechanism is individually rational if it always selects an individually rational matching. Similarly, one can speak of non wasteful, stable, or Pareto-efficient mechanisms. Finally, a mechanism is *strategy-proof* if no student can ever benefit by unilaterally misrepresenting his preferences.<sup>10</sup>

### 3 Three Competing Mechanisms

In this section we describe the mechanisms that we study in the context of constrained school choice: the Boston mechanism, the Gale-Shapley Student-Optimal Stable mechanism, and the Top Trading Cycles mechanism. The three mechanisms are direct mechanisms, *i.e.*, students only need to report an ordered list of their acceptable schools. For a profile of revealed preferences the matching that is selected by a mechanism is computed via an algorithm. Below we give a description of the three algorithms. Let  $(I, S, q, P, f)$  be a school choice problem.

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<sup>10</sup>In game theoretic terms, a mechanism is strategy-proof if truthful preference revelation is a weakly dominant strategy.

### 3.1 The Boston Algorithm

The Boston algorithm was first described in the literature by Abdulkadiroğlu and Sönmez (2003). Consider a profile of ordered lists  $Q$  submitted by the students. The Boston algorithm finds a matching through the following steps.

STEP 1: Set  $q_s^1 := q_s$  for all  $s \in S$ . Each student  $i$  proposes to the school that is ranked first in  $Q_i$  (if there is no such school then  $i$  remains unassigned). Each school  $s$  assigns up to  $q_s^1$  seats to its proposers one at a time following the priority order  $f_s$ . Remaining students are rejected. Let  $q_s^2$  denote the number of available seats at school  $s$ . If  $q_s^2 = 0$  then school  $s$  is removed.

STEP  $l$ ,  $l \geq 2$ : Each student  $i$  that is rejected in Step  $l - 1$  proposes to the next school in the ordered list  $Q_i$  (if there is no such school then  $i$  remains unassigned). School  $s$  assigns up to  $q_s^l$  seats to its (new) proposers one at a time following the priority order  $f_s$ . Remaining students are rejected. Let  $q_s^l$  denote the number of available seats at school  $s$ . If  $q_s^l = 0$  then school  $s$  is removed.

The algorithm stops when no student is rejected or all schools have been removed. Any remaining student remains unassigned. Let  $\beta(Q)$  denote the matching. The mechanism  $\beta$  is the *Boston mechanism*. It is well known that the Boston mechanism is individually rational, non wasteful, and Pareto-efficient. It is, however, neither stable nor strategy-proof.

### 3.2 The Gale-Shapley Deferred Acceptance (DA) Algorithm

The deferred acceptance algorithm was introduced by Gale and Shapley (1962). Let  $Q$  be a profile of ordered lists submitted by the students. The DA algorithm finds a matching through the following steps.

STEP 1: Each student  $i$  proposes to the school that is ranked first in  $Q_i$  (if there is no such school then  $i$  remains unassigned). Each school  $s$  tentatively assigns up to  $q_s$  seats to its proposers one at a time following the priority order  $f_s$ . Remaining students are rejected.

STEP  $l$ ,  $l \geq 2$ : Each student  $i$  that is rejected in Step  $l - 1$  proposes to the next school in the ordered list  $Q_i$  (if there is no such school then  $i$  remains unassigned). Each school  $s$  considers the new proposers and the students that have a (tentative) seat at  $s$ . School  $s$  tentatively assigns up to  $q_s$  seats to these students one at a time following the priority order  $f_s$ . Remaining students are rejected.

The algorithm stops when no student is rejected. Each student is assigned to his final tentative school. Let  $\gamma(Q)$  denote the matching. The mechanism  $\gamma$  is the *Student-Optimal Stable mechanism*. The Student-Optimal Stable mechanism is a stable mechanism that is Pareto superior to any other stable matching mechanism (Gale and Shapley, 1962). An additional important property of the Student-Optimal Stable mechanism is that it is strategy-proof (Dubins and Freedman, 1981; Roth, 1982). However, it is not Pareto-efficient.

### 3.3 The Top Trading Cycles (TTC) Algorithm

The Top Trading Cycles mechanism in the context of school choice was introduced by Abdulkadiroğlu and Sönmez (2003).<sup>11</sup> Let  $Q$  be a profile of ordered lists submitted by the students. The TTC algorithm finds a matching through the following steps.

STEP 1: Set  $q_s^1 := q_s$  for all  $s \in S$ . Each student  $i$  points to the school that is ranked first in  $Q_i$  (if there is no such school then  $i$  points to himself, *i.e.*, he forms a *self-cycle*). Each school  $s$  points to the student that has the highest priority in  $f_s$ . There is at least one cycle. If a student is in a cycle he is assigned a seat at the school he points to (or to himself if he is in a self-cycle). Students that are assigned are removed. If a school  $s$  is in a cycle and  $q_s^1 = 1$ , then the school is removed. If a school  $s$  is in a cycle and  $q_s^1 > 1$ , then the school is not removed and its capacity becomes  $q_s^2 := q_s^1 - 1$ .

STEP  $l$ ,  $l \geq 2$ : Each student  $i$  that is rejected in Step  $l - 1$  points to the next school in the ordered list  $Q_i$  that has not been removed at some step  $r$ ,  $r < l$ , or points to himself if there is no such school. Each school  $s$  points to the student with the highest priority in  $f_s$  among the students that have not been removed at a step  $r$ ,  $r < l$ . There is at least one cycle. If a student is in a cycle he is assigned a seat at the school he points to (or to himself if he is in a self-cycle). Students that are assigned are removed. If a school  $s$  is in a cycle and  $q_s^l = 1$ , then the school is removed. If a school  $s$  is in a cycle and  $q_s^l > 1$ , then the school is not removed and its capacity becomes  $q_s^{l+1} := q_s^l - 1$ .

The algorithm stops when all students or all schools have been removed. Any remaining student is assigned to himself. Let  $\tau(Q)$  denote the matching. The mechanism  $\tau$  is the *Top Trading Cycles mechanism*. The Top Trading Cycles mechanism is a Pareto-efficient

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<sup>11</sup>The Top Trading Cycles mechanism was inspired by Gale's Top Trading Cycles algorithm which was used by Roth and Postlewaite (1977) to obtain the unique core allocation for housing markets (Shapley and Scarf, 1974). A variant of the Top Trading Cycles mechanism was introduced by Abdulkadiroğlu and Sönmez (1999) for a model of house allocation with existing tenants.

and strategy-proof mechanism (see Abdulkadiroğlu and Sönmez, 2003, for proofs in the context of school choice). The mechanism is also individually rational and non wasteful. However, it is not stable.

### 3.4 An Illustrative Example

We illustrate the working of the three mechanisms in the following example.

Let  $I = \{i_1, i_2, i_3, i_4\}$  be the set of students,  $S = \{s_1, s_2, s_3\}$  be the set of schools, and  $q = (1, 2, 1)$  be the capacity vector. The students' preferences  $P$  and the priority structure  $f$  are given in the table below. So, for instance,  $P_{i_1} = s_2, s_1$  and  $f_{s_1}(i_1) < f_{s_1}(i_2) < f_{s_1}(i_3) < f_{s_1}(i_4)$ .

$P_{i_1}$	$P_{i_2}$	$P_{i_3}$	$P_{i_4}$	$f_{s_1}$	$f_{s_2}$	$f_{s_3}$
$s_2$	$s_1$	$s_1$	$s_2$	$i_1$	$i_3$	$i_4$
$s_1$	$s_2$	$s_2$	$s_3$	$i_2$	$i_4$	$i_1$
	$s_3$		$s_1$	$i_3$	$i_1$	$i_2$
				$i_4$	$i_2$	$i_3$

If the students truthfully report their preference lists, then the mechanisms yield the following matchings.

*The Boston mechanism.*

**Step 1.** Each student proposes to his most preferred school. So, school  $s_1$  receives a proposal from  $i_2$  and  $i_3$ . Student  $i_2$  has a higher priority, so  $i_3$ 's proposal is rejected and  $i_2$  is assigned the unique seat at  $s_1$ . School  $s_2$  receives a proposal from  $i_1$  and  $i_4$ . Since school  $s_2$  has 2 seats each of the students  $i_1$  and  $i_4$  is assigned a seat at  $s_2$ . Schools  $s_1$  and  $s_2$  have filled all their seats and hence are removed. The tentative matching is  $\{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}$ .

**Step 2.** Student  $i_3$  cannot propose to his next preferred school,  $s_2$ . Since he finds school  $s_3$  unacceptable he is removed and remains unassigned. So, the final matching is given by

$$\beta(P) = \{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}.$$

*The Student-Optimal Stable mechanism.*

**Step 1.** Each student proposes to his most preferred school. So, school  $s_1$  receives a proposal from  $i_2$  and  $i_3$ . Student  $i_2$  has a higher priority, so  $i_3$ 's proposal is rejected. School  $s_2$  receives a proposal from  $i_1$  and  $i_4$ . Since school  $s_2$  has 2 seats it does not reject any of the two students. The tentative matching is  $\{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}$ .

**Step 2.** Student  $i_3$  proposes to school  $s_2$ . So, now school  $s_2$  has two (tentatively) accepted students,  $i_1$  and  $i_4$ , and one new proposal, from  $i_3$ . Since school  $s_2$  has 2 seats it rejects  $i_1$ , the student with the lowest priority. The tentative matching becomes  $\{\{s_1, i_2\}, \{s_2, i_3, i_4\}, \{s_3\}, \{i_1\}\}$ .

**Step 3.** Student  $i_1$  proposes to school  $s_1$ . The unique seat of school  $s_1$  is tentatively occupied by  $i_2$ . Since  $i_1$  has a higher priority than student  $i_2$ , the latter is rejected. The tentative matching becomes  $\{\{s_1, i_1\}, \{s_2, i_3, i_4\}, \{s_3\}, \{i_2\}\}$ .

**Step 4.** Student  $i_2$  proposes to school  $s_3$ . Since school  $s_3$ 's unique seat is available, student  $i_2$  is accepted. No student has been rejected in this step, so the tentative matching is the final matching and is given by

$$\gamma(P) = \{\{s_1, i_1\}, \{s_2, i_3, i_4\}, \{s_3, i_2\}\}.$$

*The Top Trading Cycles mechanism.*

**Step 1.** Each student points to his most preferred school, and each school points to the student with highest priority. There is a unique cycle that is given by  $(i_1, s_2, i_3, s_1)$ . So, students  $i_1$  and  $i_3$  are assigned a seat at schools  $s_2$  and  $s_1$ , respectively. Students  $i_1$  and  $i_3$  are removed. Since school  $s_1$  had only 1 available seat it is also removed. School  $s_2$  still has an available seat and is therefore not removed. The tentative matching is  $\{\{s_1, i_3\}, \{s_2, i_1\}, \{s_3\}, \{i_2\}, \{i_4\}\}$ .

**Step 2.** There is a unique cycle given by  $(i_4, s_2)$ . So, student  $i_4$  is assigned the remaining seat at school  $s_2$ . Both student  $i_4$  and school  $s_2$  are removed. The tentative matching is  $\{\{s_1, i_3\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_2\}\}$ .

**Step 3.** Only student  $i_2$  and school  $s_3$  remain. Since  $i_2$  finds school  $s_3$  acceptable, he points to the school. Since  $i_2$  is the only remaining student, school  $s_3$  points to  $i_2$ . This creates a cycle and hence  $i_2$  is assigned a seat at school  $s_3$ . So, the final matching is

$$\tau(P) = \{\{s_1, i_3\}, \{s_2, i_1, i_4\}, \{s_3, i_2\}\}.$$

Note that for the school choice problem above the three mechanisms generate different matchings. Also, the obtained matchings illustrate directly some of the “problems” of the mechanisms. For instance,  $\beta(P)$  is Pareto-efficient but not stable because student  $i_3$  has justified envy with respect to school  $s_2$  and any of the students that occupy a seat. In fact, one readily sees that  $\beta$  is not strategy-proof. (Had student  $i_3$  have announced the list that only contains school  $s_2$  he would have guaranteed a seat at this school.) Similarly, one easily verifies that  $\gamma(P)$  is stable but not Pareto-efficient and that  $\tau(P)$  is Pareto-efficient but not stable. More importantly, note that if in a direct revelation game under  $\gamma$  or  $\tau$

students could only submit a list of 2 schools, student  $i_2$  would remain unassigned (and the other students unaffected), provided that each student submits the list with his two most preferred schools. Therefore, if students can only submit short preference lists, then (at least) student  $i_2$  ought to strategize to ensure a seat at some (acceptable) school.

## 4 Constrained Preference Revelation

Fix the priority ordering  $f$  and the capacities  $q$ . We consider the following school choice procedure. Students are asked to submit (simultaneously) preference lists  $Q = (Q_{i_1}, \dots, Q_{i_n})$  of “length” at most  $k$  (i.e., preference lists with at most  $k$  acceptable schools). Here,  $k$  is a positive integer,  $1 \leq k \leq m$ , and is called the *quota*. Subsequently, a mechanism  $\varphi$  is used to obtain the matching  $\varphi(Q)$  and for all  $i \in I$ , student  $i$  is assigned a seat at school  $\varphi(Q)(i)$ .

Clearly, the above procedure induces a strategic form game, the *quota-game*  $\Gamma^\varphi(P, k) := \langle I, \mathcal{Q}(k)^I, P \rangle$ . The set of players is the set of students  $I$ . The strategy set of each student is the set of preference lists with at most  $k$  acceptable schools and is denoted by  $\mathcal{Q}(k)$ . Let  $\mathcal{Q} := \mathcal{Q}(m)$ . Outcomes of the game are evaluated through the true preferences  $P = (P_{i_1}, \dots, P_{i_n})$ , where with some abuse of notation  $P$  denotes the straightforward extension of the preference relation over schools (and the option of remaining unassigned) to matchings. That is, for all  $i \in I$  and matchings  $\mu$  and  $\mu'$ ,  $\mu P_i \mu'$  if and only if  $\mu(i) P_i \mu'(i)$ .

For any profile of preferences  $Q \in \mathcal{Q}^I$  and any  $i \in I$ , we write  $Q_{-i}$  for the profile of preferences that is obtained from  $Q$  after leaving out preferences  $Q_i$  of student  $i$ . A profile of submitted preference lists  $Q \in \mathcal{Q}(k)^I$  is a Nash equilibrium of the game  $\Gamma^\varphi(P, k)$  (or *k-Nash equilibrium* for short) if for all  $i \in I$  and all  $Q'_i \in \mathcal{Q}(k)$ ,  $\varphi(Q_i, Q_{-i}) R_i \varphi(Q'_i, Q_{-i})$ . Let  $\mathcal{E}^\varphi(P, k)$  denote the set of  $k$ -Nash equilibria. Let  $\mathcal{O}^\varphi(P, k)$  denote the set of  $k$ -Nash equilibrium outcomes, i.e.,  $\mathcal{O}^\varphi(P, k) := \{\varphi(Q) : Q \in \mathcal{E}^\varphi(P, k)\}$ .

**Remark 4.1** Setting the same quota for all students is without loss of generality since in the proofs we never compare the values of the quota for different students. For the Student-Optimal Stable mechanism we will see that giving some students a higher quota can only expand the set of Nash equilibrium outcomes with unstable matchings (Proposition 6.4 and Theorem 6.5). As for the Top Trading Cycles mechanism, we will see that the set of Nash equilibrium outcomes does not vary with the quota (Theorem 7.2). Therefore, our implementation results imply that giving students different quotas cannot serve as a policy device to favor some students.

## 5 Boston Mechanism

Our first result, which will serve as a benchmark for the other two mechanisms, states that the Boston mechanism implements the set of stable matchings, independently of the quota. Note that since the set of stable matchings is always non empty (Gale and Shapley, 1962), the existence of Nash equilibria follows directly from this implementation result.

**Theorem 5.1** *For any school choice problem  $P$  and any quota  $k$ , the game  $\Gamma^\beta(P, k)$  implements  $S(P)$  in Nash equilibria, i.e.,  $\mathcal{O}^\beta(P, k) = S(P)$ .*

This result is obtained through a straightforward adaptation of the proof of Theorem 1 in Ergin and Sönmez (2006). Its proof is therefore omitted.

## 6 Student-Optimal Stable Mechanism

We first establish the existence of Nash equilibria in pure strategies when the Student-Optimal Stable mechanism is used.

**Theorem 6.1** *For any school choice problem  $P$  and any quota  $k$ ,  $\mathcal{E}^\gamma(P, k) \neq \emptyset$ .*

Theorem 6.1 is actually a direct corollary to the next result, which says that stable matchings can always be obtained as equilibrium outcomes, for any value of the quota.

**Proposition 6.2** (Romero-Medina, 1998, Theorem 7) *For any school choice problem  $P$  and any quota  $k$ ,  $S(P) \subseteq \mathcal{O}^\gamma(P, k)$ .*

One may wonder whether the converse of Proposition 6.2 holds, i.e., whether each Nash equilibrium induces a stable matching. As the following example shows, this is not the case.

**Example 6.3 An Unstable Nash Equilibrium Outcome in  $\Gamma^\gamma(P, k)$**

Let  $I = \{i_1, i_2, i_3, i_4\}$  be the set of students,  $S = \{s_1, s_2, s_3\}$  be the set of schools, and  $q = (1, 1, 1)$  be the capacity vector. The students' preferences  $P$  and the priority structure  $f$  are given in the table below. Let  $k = 2$  be the quota and  $Q \in \mathcal{Q}(2)^I$  as given below.

$P_{i_1}$	$P_{i_2}$	$P_{i_3}$	$P_{i_4}$	$f_{s_1}$	$f_{s_2}$	$f_{s_3}$				
$s_1$	$s_2$	$s_3$	$s_1$	$i_3$	$i_1$	$i_2$				
$s_2$	$s_3$	$s_1$	$s_2$	$i_1$	$i_2$	$i_4$				
$s_3$	$s_1$	$s_2$	$s_3$	$i_2$	$i_3$	$i_3$				
				$i_4$	$i_4$	$i_1$				

$Q_{i_1}$	$Q_{i_2}$	$Q_{i_3}$	$Q_{i_4}$
$s_1$	$s_2$	$s_3$	$s_1$
$s_2$	$s_3$	$s_1$	$s_2$

One easily verifies that  $\gamma(Q) = \{\{i_1, s_1\}, \{i_2, s_2\}, \{i_3, s_3\}, \{i_4\}\}$ . Since student  $i_4$  has justified envy for school  $s_3$ ,  $\gamma(Q) \notin S(P)$ . It remains to show that  $Q \in \mathcal{E}^\gamma(P, 2)$ . Since students  $i_1$ ,  $i_2$ , and  $i_3$  are assigned a seat at their favorite school, it is sufficient to check that student  $i_4$  has no profitable deviation. Notice that the only possibility for student  $i_4$  to change the outcome of the mechanism is by listing school  $s_3$ . So, the only strategies that we have to check are given by  $\bar{Q}(2) = \{Q^a, Q^b, Q^c, Q^d, Q^e\}$ , where  $Q^a = s_3$ ,  $Q^b = s_1, s_3$ ,  $Q^c = s_2, s_3$ ,  $Q^d = s_3, s_1$ , and  $Q^e = s_3, s_2$ . Routine computations show that none of these strategies is a profitable deviation. So,  $Q \in \mathcal{E}^\gamma(P, 2)$ .  $\diamond$

In light of Example 6.3, can we find a value of the quota that ensures that *all* equilibrium outcomes are stable?<sup>12</sup> The next result gives a positive answer to this question.

**Proposition 6.4** *For any school choice problem  $P$ , the game  $\Gamma^\gamma(P, 1)$  implements  $S(P)$  in Nash equilibria, i.e.,  $\mathcal{O}^\gamma(P, 1) = S(P)$ .*

Proposition 6.4 is not very surprising. When the quota is 1 the DA algorithm consists of only one step, which moreover coincides with the (then also unique step) of the Boston algorithm, i.e.,  $\Gamma^\gamma(P, 1) = \Gamma^\beta(P, 1)$ . In other words, Proposition 6.4 can be obtained as a corollary to Theorem 5.1.

If setting the quota equal to 1 allows us to implement any stable matching, what about higher values of the quota? One may well imagine that for some preference profile and other values of the quota the Student-Optimal Stable mechanism also implements the set of stable matchings. A sharp answer to this question would mostly likely lead to specific classes of preference profiles. We can, however, prove a more interesting result: The equilibria of the quota-games are nested in the sense that any  $k$ -Nash equilibrium is also a  $k'$ -Nash equilibrium where  $k'$  is greater than  $k$ . Hence, if for some value of the quota an unstable matching obtains in equilibrium then it also obtains for any higher value of the quota.

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<sup>12</sup>In fact, Lemma A.2 in the Appendix shows that all Nash equilibrium outcomes are individually rational and non wasteful. So, the question boils down to whether the equilibrium outcomes are free of justified envy.

**Theorem 6.5** For any school choice problem  $P$  and quotas  $k < k'$ ,  $\mathcal{E}^\gamma(P, k) \subseteq \mathcal{E}^\gamma(P, k')$ .

Example 6.3 and Theorem 6.5 suggest that unstable equilibrium outcomes are difficult to avoid in the quota-game that uses the Student-Optimal Stable mechanism. Hence, the only degree of freedom that is left to obtain stable equilibrium outcomes is the schools' priority structure. That is, the problem is now to see whether there exists a condition on the priority structure under which the Student-Optimal Stable mechanism implements the correspondence of stable matchings in Nash equilibria. As we show below, such a condition exists and is known as acyclicity.

**Definition 6.6 Ergin-Acyclicity** (Ergin, 2002)<sup>13</sup>

Given a priority structure  $f$ , an *Ergin-cycle* is constituted of distinct  $s, s' \in S$  and  $i, j, l \in I$  such that the following two conditions are satisfied:

*Ergin-cycle condition:*  $f_s(i) < f_s(j) < f_s(l)$  and  $f_{s'}(l) < f_{s'}(i)$  and

*ec-scarcity condition:* there exist (possibly empty and) disjoint sets  $I_s, I_{s'} \subseteq I \setminus \{i, j, l\}$  such that  $I_s \subseteq U_s^f(j)$ ,  $I_{s'} \subseteq U_{s'}^f(i)$ ,  $|I_s| = q_s - 1$ , and  $|I_{s'}| = q_{s'} - 1$ .

A priority structure is *Ergin-acyclic* if no Ergin-cycles exist. △

**Theorem 6.7** Let  $k \neq 1$ . Then,  $f$  is an Ergin-acyclic priority structure if and only if for any school choice problem  $P$ , the game  $\Gamma^\gamma(P, k)$  implements  $S(P)$  in Nash equilibria, i.e.,  $\mathcal{O}^\gamma(P, k) = S(P)$ .

Ergin (2002) showed that Ergin-acyclicity of the priority structure is necessary and sufficient for the Pareto-efficiency of the Student-Optimal Stable mechanism.<sup>14</sup> Theorem 6.7 shows that Ergin-acyclicity has a different impact depending on whether one considers the Student-Optimal Stable mechanism *per se* or in the context of the induced preference revelation game. In the former case Ergin-acyclicity induces Pareto-efficiency while in the latter case it leads to stability.

## 7 Top Trading Cycles Mechanism

Like for the Student-Optimal Stable Mechanism, we first state the existence of  $k$ -Nash equilibria for any value of the quota. Yet, for its proof we first establish several results

<sup>13</sup>Ergin (2002) used the terminology of cycles and acyclicity. However, since we will need to introduce another acyclicity concept due to Kesten (2006) we conveniently change the terminology.

<sup>14</sup>Ergin (2002) also showed that Ergin-acyclicity is sufficient for group strategy-proofness and consistency of the Student-Optimal Stable mechanism as well as necessary for each of these conditions separately.

about the structure of the set of Nash equilibria in order to circumvent certain difficulties.<sup>15</sup>

**Theorem 7.1** *For any school choice problem  $P$  and quota  $k$ ,  $\mathcal{E}^\tau(P, k) \neq \emptyset$ .*

Very similarly to the Boston mechanism, the Top Trading Cycle mechanism was initially not introduced to produce stable matchings. Nevertheless, one may wonder whether the equilibrium outcomes of the induced preference revelation game are stable. The main reason to study this question is that both the Boston mechanism and the Student-Optimal Stable mechanism perform differently as a mechanism *per se* or in the context of the induced preference revelation game. In other words, *a priori* there is no reason to suspect that the Top Trading Cycle mechanism is unable to produce stable matchings in equilibria. Before considering the stability properties of the equilibrium outcomes under the Top Trading Cycle mechanism we first establish a major result concerning the set of Nash equilibrium outcomes.

**Theorem 7.2** *For any school choice problem  $P$  and quota  $k$ ,  $\mathcal{O}^\tau(P, k) = \mathcal{O}^\tau(P, 1)$ .*

Theorem 7.2 allows us to greatly simplify our analysis of the Nash equilibria. Indeed, if a matching  $\mu$  obtains in equilibrium we can use Theorem 7.2 to deduce, to some extent, the strategies used by the students. More precisely, if in an equilibrium the matching  $\mu$  obtains then we can deduce that there exists an equilibrium in which each student  $i \in I$  uses the strategy  $Q_i = \mu(i)$ . In fact, we prove in the Appendix a stronger result than Theorem 7.2. We indeed show that if we consider a  $k$ -Nash equilibrium  $Q$ , then restricting the strategy of any student  $i \in I$  to only one element,  $\tau(Q)(i)$  and leaving unchanged the other students' strategies also constitutes a  $k$ -Nash equilibrium. That is, if  $Q$  is a  $k$ -Nash equilibrium then for any student  $i \in I$ ,  $(\tau(Q)(i), Q_{-i})$  is also a  $k$ -Nash equilibrium.

As we show below, Theorem 7.2 proves to be useful to study the relation between equilibrium outcomes and the set of stable matchings. The following example already suggests that regarding stability in equilibrium, the Top Trading Cycle mechanism performs worse than the Student-Optimal Stable mechanism.<sup>16</sup>

**Example 7.3 A School Choice Problem  $P$  with  $S(P) \cap \mathcal{O}^\tau(P, 1) = \emptyset$**

Let  $I = \{i_1, i_2, i_3\}$  be the set of students,  $S = \{s_1, s_2\}$  be the set of schools, and  $q = (1, 1)$

<sup>15</sup>The complexity of the TTC algorithm makes it indeed difficult to visualize the collection of new cycles that may arise when a student deviates from a particular strategy profile.

<sup>16</sup>Using Theorem 7.2 one can nevertheless show that all Nash equilibrium outcomes are again individually rational and non wasteful. The proof is available from the authors upon request.

be the capacity vector. The students' preferences  $P$  and the priority structure  $f$  are given in the table below.

$P_{i_1}$	$P_{i_2}$	$P_{i_3}$	$f_{s_1}$	$f_{s_2}$
$s_2$	$s_1$	$s_1$	$i_1$	$i_3$
			$i_2$	$i_2$
			$i_3$	$i_1$

It is easy to check that the unique stable matching is  $\mu = \{\{i_1, s_2\}, \{i_2, s_1\}, \{i_3\}\}$ . We show that  $\mu$  cannot be sustained at any Nash equilibrium of the game  $\Gamma^\tau(P, 1)$ . Suppose to the contrary that  $\mu$  can be sustained at some Nash equilibrium. In other words, there is a profile  $Q \in \mathcal{Q}(1)^I$  such that  $\tau(Q) = \mu$  and  $Q \in \mathcal{E}^\tau(P, 1)$ . Since  $\tau(Q) = \mu$ ,  $Q_{i_1} = s_2$  and  $Q_{i_2} = s_1$ . If  $Q_{i_3} = s_1$ , then  $\tau(Q)(i_3) = s_1 \neq \mu(i_3)$ . So,  $Q_{i_3} \neq s_1$ . But then  $\tau(Q')P_{i_3}\tau(Q)$  for  $Q' := (Q_{i_1}, Q_{i_2}, s_1)$ . Hence,  $Q \notin \mathcal{E}^\tau(P, 1)$ , a contradiction.  $\diamond$

Since there is no hope to choose an adequate level of the quota to ensure the stability of the equilibrium outcomes under the Top Trading Cycle mechanism, we turn to the second dimension of the mechanism, *i.e.*, the schools' priority structure. The issue here is similar to that of the Student-Optimal Stable mechanism, *i.e.*, to see whether there exists a condition on the priority structure that would ensure the stability of any equilibrium outcome. Indeed, the last result of this section is that Kesten's (2006) acyclicity condition is necessary and sufficient for the Top Trading Cycles mechanism to implement the correspondence of stable matchings in Nash equilibria.

**Definition 7.4 Kesten-Acyclicity** (Kesten, 2006)

Given a priority structure  $f$ , a *Kesten-cycle* is constituted of distinct  $s, s' \in S$  and  $i, j, l \in I$  such that the following two conditions are satisfied:

*Kesten-cycle condition*  $f_s(i) < f_s(j) < f_s(l)$  and  $f_{s'}(l) < f_{s'}(i), f_{s'}(j)$  and

*kc-scarcity condition* there exists a (possibly empty) set  $I_s \subseteq I \setminus \{i, j, l\}$  with  $I_s \subseteq U_s^f(i) \cup [U_s^f(j) \setminus U_{s'}^f(l)]$  and  $|I_s| = q_s - 1$ .

A priority structure is *Kesten-acyclic* if no Kesten-cycles exist.  $\triangle$

Kesten (2006) showed that Kesten-acyclicity of the priority structure is necessary and sufficient for the stability of the Top Trading Cycles mechanism when students report their true preferences.<sup>17</sup> Kesten-acyclicity implies Ergin-acyclicity (Lemma 1, Kesten, 2006).

<sup>17</sup>Kesten (2006) also showed that Kesten-acyclicity is necessary and sufficient for the Top Trading Cycles mechanism to be resource monotonic and population monotonic. In addition, he also proved that the Top Trading Cycles mechanism coincides with the Student-Optimal Stable mechanism if and only if the priority structure is Kesten-acyclic.

It is easy to check that the reverse holds if all schools have capacity 1.

**Theorem 7.5** *Let  $1 \leq k \leq m$ . Then,  $f$  is a Kesten-acyclic priority structure if and only if for any school choice problem  $P$ , the game  $\Gamma^\tau(P, k)$  implements  $S(P)$  in Nash equilibria, i.e.,  $\mathcal{O}^\tau(P, k) = S(P)$ .*

Kesten’s (2006) result and Theorem 7.5 have in common that Kesten-acyclicity is both necessary and sufficient for the stability of the Top Trading Cycle mechanism. Yet, it is important to note that, contrary to Kesten (2006), in our game students typically cannot reveal their true preferences.

## 8 Equilibria in Truncations

In this section we focus on “truncation” strategies which are shown to be undominated in the quota-games induced by both the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism. We first strengthen the negative side of Theorems 6.7 and 7.5 by showing that the strategy profile of Example 6.3 is in fact a strong Nash equilibrium in truncations that induces an unstable matching. Next, again for both mechanisms, we will show that in general there is also no relation between the set of unassigned students at equilibrium and the set of unassigned students in stable matchings. However, for Nash equilibria in truncations we do obtain a positive result in this respect for the Student-Optimal Stable mechanism.

One piece of advice about which preference list a student should submit follows from the strategy-proofness of the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism in the unconstrained setting: it does not pay off to submit a list of schools that does not respect the true order. More precisely, a list that does not respect the order of a student’s true preferences is weakly dominated by listing the same schools in the “true order.” Let  $\varphi$  be a mechanism. Student  $i$ ’s strategy  $Q_i \in \mathcal{Q}(k)$  in the game  $\Gamma^\varphi(P, k)$  is *weakly  $k$ -dominated* by another strategy  $Q'_i \in \mathcal{Q}(k)$  if  $\varphi(Q'_i, Q_{-i}) R_i \varphi(Q_i, Q_{-i})$  for all  $Q_{-i} \in \mathcal{Q}(k)^{I \setminus i}$ .

**Lemma 8.1** *Let  $P$  be a school choice problem. Let  $1 \leq k \leq m$ . Let  $i \in I$  be a student. Consider two strategies  $Q_i, Q'_i \in \mathcal{Q}(k)$  such that (a)  $Q_i$  and  $Q'_i$  contain the same set of schools, and (b) for any two schools  $s$  and  $s'$  listed in  $Q_i$  (or  $Q'_i$ ),  $sQ'_i s'$  implies  $sP_i s'$ . Then,  $Q_i$  is weakly  $k$ -dominated by  $Q'_i$  in the games  $\Gamma^\gamma(P, k)$  and  $\Gamma^\tau(P, k)$ .*

The message of Lemma 8.1 is clear: a student cannot lose (and may possibly gain) by submitting the same set of schools in the true order. A special type of strategies that satisfy this condition are the so-called truncations. A *truncation* of a preference list  $P_i$  is a list  $P'_i$  obtained from  $P_i$  by deleting some school and all less preferred acceptable schools. The following lemma says that in the games  $\Gamma^\gamma(P, k)$  and  $\Gamma^\tau(P, k)$  submitting a truncation “as long as possible” is  $k$ -undominated. Formally, student  $i$ ’s strategy  $Q_i \in \mathcal{Q}(k)$  is  $k$ -dominated by another strategy  $Q'_i \in \mathcal{Q}(k)$  if  $\varphi(Q'_i, Q_{-i})R_i\varphi(Q_i, Q_{-i})$  for all  $Q_{-i} \in \mathcal{Q}(k)^{\wedge i}$  and  $\varphi(Q'_i, Q'_{-i})P_i\varphi(Q_i, Q'_{-i})$  for some  $Q'_{-i} \in \mathcal{Q}(k)^{\wedge i}$ . A strategy in  $\mathcal{Q}(k)$  is  $k$ -undominated if it is not  $k$ -dominated by any other strategy in  $\mathcal{Q}(k)$ .

**Lemma 8.2** *Let  $P$  be a school choice problem. Let  $1 \leq k \leq m$ . Let  $i \in I$  be a student. Denote the number of (acceptable) schools in  $P_i$  by  $|P_i|$ . Then, the strategy  $P_i^k$  of submitting the first  $\min\{k, |P_i|\}$  schools of the true preference list  $P_i$  in the true order is  $k$ -undominated in the games  $\Gamma^\gamma(P, k)$  and  $\Gamma^\tau(P, k)$ .*

Although the strategy profile  $P^k := (P_i^k)_{i \in I}$  is a profile of  $k$ -undominated strategies, it is not necessarily a Nash equilibrium in the games  $\Gamma^\gamma(P, k)$  and  $\Gamma^\tau(P, k)$ . In case it is a Nash equilibrium it may still induce an unstable matching as Example 6.3 shows.

**Example 6.3 continued. For both  $\gamma$  and  $\tau$ : A Strong Nash Equilibrium in (Undominated) Truncations that yields an Unstable Matching**

Consider again the strategy profile  $Q = P^2 \in \mathcal{Q}(2)^I$  of 2-undominated truncations. Since students  $i_1, i_2$ , and  $i_3$  are assigned a seat at their favorite school at  $\gamma(Q)$  and  $Q \in \mathcal{E}^\gamma(P, 2)$ , it follows that  $Q$  is a strong Nash equilibrium (*cf.* Aumann, 1959) in  $\Gamma^\gamma(P, 2)$ .

As for the Top Trading Cycles mechanism, one easily verifies that also  $\tau(Q) = \{\{i_1, s_1\}, \{i_2, s_2\}, \{i_3, s_3\}, \{i_4\}\}$ . For the same reason as before, it is sufficient to check that student  $i_4$  has no profitable deviation. This, however, is immediate since student  $i_4$  cannot “break” the cycle  $(i_1, s_1, i_3, s_3, i_2, s_2)$  that forms in the first step of the TTC algorithm. Hence,  $Q$  is also a strong Nash equilibrium in  $\Gamma^\tau(P, 2)$ .  $\diamond$

The results of McVitie and Wilson (1970) and Roth (1984b) for college admissions imply that for any school choice problem the set of unassigned students is the same for all stable matchings.<sup>18</sup> In other words, for  $\mu, \mu' \in S(P)$ ,  $\mu(i) = i$  implies  $\mu'(i) = i$ . Given the re-

<sup>18</sup>A generalization of this result is known in the two-sided matching literature as the “Rural Hospital Theorem” (Roth, 1986) and says that the degree of occupation and quality of interns at typically less demanded rural hospitals in the US is not due to the choice of a specific stable matching.

strictiveness of the acyclicity conditions to guarantee stable Nash equilibrium outcomes, one may wonder whether at least always the set of unassigned students at equilibrium coincides with the set of unassigned students in stable matchings. In fact, a less ambitious idea would be to establish that at equilibrium the *number* of unassigned students equals the number of unassigned students in stable matchings. The following two examples show that in general this is not true. In other words, the number of unassigned students at equilibrium is not inherited from that of the set of stable matchings. Given Proposition 6.2, this in particular implies for the Student-Optimal Stable mechanism that the number of unassigned students can vary from one equilibrium outcome to another.

**Example 8.3 For both  $\gamma$  and  $\tau$ : Less Assigned Students in an Equilibrium than in Stable Matchings**

Let  $I = \{i_1, i_2, i_3\}$  be the set of students,  $S = \{s_1, s_2, s_3\}$  be the set of schools, and  $q = (1, 1, 1)$  be the capacity vector. The students' preferences  $P$  and the priority structure  $f$  are given in the table below. One easily verifies that strategy profile  $Q$  given below is a Nash equilibrium in  $\Gamma^\gamma(P, 2)$  and  $\Gamma^\tau(P, 2)$ .

$P_{i_1}$	$P_{i_2}$	$P_{i_3}$	$f_{s_1}$	$f_{s_2}$	$f_{s_3}$			
$s_1$	$s_3$	$s_3$	$i_3$	$i_2$	$i_1$			
$s_3$	$s_1$	$s_2$	$i_1$	$i_3$	$i_2$			
$s_2$		$s_1$	$i_2$	$i_1$	$i_3$			

  

$Q_{i_1}$	$Q_{i_2}$	$Q_{i_3}$
$s_1$	$s_1$	$s_3$
$s_3$		$s_1$

Since  $\gamma(Q) = \tau(Q) = \{\{i_1, s_1\}, \{i_3, s_3\}, \{i_2\}, \{s_2\}\}$  and  $\gamma(P) = \{\{i_1, s_1\}, \{i_2, s_3\}, \{i_3, s_2\}\}$ , there are less assigned students at  $\gamma(Q) = \tau(Q)$  than in any stable matching.  $\diamond$

**Example 8.4 For both  $\gamma$  and  $\tau$ : More Assigned Students in an Equilibrium than in Stable Matchings**

Let  $I = \{i_1, i_2, i_3\}$  be the set of students,  $S = \{s_1, s_2, s_3\}$  be the set of schools, and  $q = (1, 1, 1)$  be the capacity vector. The students' preferences  $P$  and the priority structure  $f$  are given in the table below. One easily verifies that strategy profile  $Q$  given below is a Nash equilibrium in  $\Gamma^\gamma(P, 2)$  and  $\Gamma^\tau(P, 2)$ .

$P_{i_1}$	$P_{i_2}$	$P_{i_3}$	$f_{s_1}$	$f_{s_2}$	$f_{s_3}$			
$s_2$	$s_3$	$s_3$	$i_3$	$i_2$	$i_1$			
	$s_2$	$s_2$	$i_1$	$i_3$	$i_2$			
	$s_1$	$s_1$	$i_2$	$i_1$	$i_3$			

  

$Q_{i_1}$	$Q_{i_2}$	$Q_{i_3}$
$s_2$	$s_3$	$s_1$
$s_3$	$s_2$	$s_2$

Since  $\gamma(Q) = \tau(Q) = \{\{i_1, s_2\}, \{i_2, s_3\}, \{i_3, s_1\}\}$  and  $\gamma(P) = \{\{i_2, s_3\}, \{i_3, s_2\}, \{i_1\}, \{s_1\}\}$ , there are more assigned students at  $\gamma(Q) = \tau(Q)$  than in any stable matching.  $\diamond$

We do obtain a positive result for  $\gamma$  if we restrict ourselves to equilibria in truncations. More precisely, the following proposition says that if a profile of truncations is a Nash equilibrium in the game  $\Gamma^\gamma(P, k)$  then the set of assigned students at the equilibrium coincides with the set of assigned students at any stable matching. In fact, each Nash equilibrium in truncations in the game  $\Gamma^\gamma(P, k)$  yields a matching that is either the student-optimal stable matching  $\gamma(P)$  or Pareto dominates  $\gamma(P)$ . For a matching  $\mu$ , denote  $M(\mu)$  for the set of assigned students, *i.e.*,  $M(\mu) := \{i \in I : \mu(i) \neq i\}$ .

**Proposition 8.5** *Let  $P$  be a school choice problem. Let  $1 \leq k \leq m$ . If  $P^k \in \mathcal{E}^\gamma(P, k)$ , then  $M(\gamma(P^k)) = M(\gamma(P))$ . In fact,  $\gamma(P^k) R_i \gamma(P)$  for all  $i \in I$ .*

For  $\tau$  we cannot obtain a similar result as the following proposition shows.

**Proposition 8.6** *Let  $P$  be a school choice problem. Let  $1 \leq k \leq m$ . If  $P^k \in \mathcal{E}^\tau(P, k)$ , then possibly  $|M(\tau(P^k))| < |M(\gamma(P))|$  or  $|M(\tau(P^k))| > |M(\gamma(P))|$ .*

## 9 Discussion

Our results show that in all three school choice procedures stability can be guaranteed by strategic interaction in spite of possible constraints on preference revelation. While no particular assumption is needed for the Boston mechanism, stringent conditions are required for the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism, namely Ergin-acyclicity and Kesten-acyclicity. Since real-life priority structures typically do not satisfy these conditions, the transition to either the Student-Optimal Stable mechanism or the Top Trading Cycles mechanism ought to come with a higher risk that students seek legal action as lower priority students may occupy more preferred schools. On the other hand, asking students to submit a list containing all their acceptable schools can be very demanding in the case of a school district with a large number of schools. Indeed, there is now a large literature showing that individuals' decisions can be drastically affected when the size of the choice set varies.<sup>19</sup> One way out could be to allow students to submit preference lists in the same way as school priorities are very often constructed in real-life, *i.e.*, allowing for indifference classes. In this situation parents would be asked to submit a ranking of schools and/or sets of schools. A random device to break ties between schools in the same equivalence class would give the regulator a strict ordering of all acceptable schools for each participant.

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<sup>19</sup>See for instance Iyengar and Kamenica (2006).

Besides possible policy implications, our results also illuminate the importance of the acyclicity conditions due to Ergin (2002) and Kesten (2006). We moreover contribute to the theory of implementation in matching markets. To the best of our knowledge, the current paper provides the first complete analysis of the equilibria in the preference revelation game induced by the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism. All previous studies, except Romero-Medina (1998), assumed students' preference revelation to be unconstrained. Given the strategy-proofness of both mechanisms in the unconstrained setting, an analysis of all (other) equilibria was therefore in some sense unnecessary. It is well-known that in the context of two-sided matching, preference revelation induced by stable mechanisms may have unstable equilibrium outcomes (*cf.* Alcalde, 1996 and Sönmez, 1997).<sup>20</sup> In the context of school choice, where only one side of the market is strategic, Ergin and Sönmez (2006) showed that this negative result can be avoided by using the Boston mechanism. We show that in this sense also the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism can be employed, as long as the priority structure is acyclic.

## A Appendix: Student-Optimal Stable Mechanism, Proofs

Let  $Q \in \mathcal{Q}^I$ . We denote  $DA(Q)$  for the application of the DA algorithm (with students proposing) to  $Q$ .

Unlike the second part, the first and relevant part of the proof of Theorem 7 in Romero-Medina (1998), which implies Proposition 6.2, is flawless. Nevertheless, we provide, for the sake of completeness, a (more formal) proof of Proposition 6.2.

**Proof of Proposition 6.2** Let  $\mu \in S(P)$ . Define  $Q_i := \mu(i) \in \mathcal{Q}(k)$  for all  $i \in I$ . Then,  $|\{i \in I : Q_i = s\}| = |\mu(s)| \leq q_s$  for all  $s \in S$ . It follows that in the first step of  $DA(Q)$  no student is rejected. So,  $\gamma(Q) = \mu$ . It remains to prove that  $Q \in \mathcal{E}^\gamma(P, k)$ . Suppose to the contrary that  $Q \notin \mathcal{E}^\gamma(P, k)$ . Then, there exists a student  $i$  and a strategy  $Q'_i \in \mathcal{Q}(k)$  such that  $\gamma(Q'_i, Q_{-i})P_i\gamma(Q) = \mu$ . Since  $\gamma(Q) = \mu \in S(P)$ ,  $\gamma(Q) \in IR(P)$ . Hence,  $\gamma(Q'_i, Q_{-i})(i) \in S$ . Denote  $s = \gamma(Q'_i, Q_{-i})(i)$ . Note  $i \notin \mu(s)$ . Consider  $DA(Q'_i, Q_{-i})$ . Of the students in  $I \setminus i$ , only the students in  $\mu(s)$  make their unique proposal to  $s$ ; all other students make either a unique proposal to another school or make no proposal at

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<sup>20</sup>Other recent papers on implementation in various settings of two-sided matching include Pais (2006), Shinotsuka and Takamiya (2003), Sotomayor (2003), and Suh (2003).

all. Since  $\gamma(Q'_i, Q_{-i})(i) = s$ , it follows that student  $i$  starts making proposals but gets rejected until he proposes to  $s$  and gets assigned a seat at  $s$  (now the DA algorithm ends since no new proposals are made). Since under  $(Q'_i, Q_{-i})$  school  $s$  accepts  $i$  it must be that  $|\mu(s)| < q_s$  or there is a student  $j \in \mu(s)$  with  $f_s(j) > f_s(i)$ . In the first case,  $\mu$  is wasteful for  $P$ , contradicting  $\mu \in S(P)$ . In the second case, student  $i$  has justified envy at  $\mu$ , also contradicting  $\mu \in S(P)$ . So,  $Q \in \mathcal{E}^\gamma(P, k)$ . ■

**Proof of Proposition 6.4** Follows from Theorem 5.1 and  $\Gamma^\gamma(P, 1) = \Gamma^\beta(P, 1)$ . ■

We will make use of the following two results to prove Theorem 6.5.

**Lemma A.1** (Roth, 1982, Lemma 1; cf. Roth and Sotomayor 1990, Lemma 4.8)

Let  $Q \in \mathcal{Q}^I$  and  $i \in I$ . Let  $Q'_i \in \mathcal{Q}$  be a preference list whose first choice is  $\gamma(Q)(i)$  if  $\gamma(Q)(i) \neq i$ , and the empty list otherwise. Then,  $\gamma(Q'_i, Q_{-i})(i) = \gamma(Q)(i)$ .

**Lemma A.2** For any school choice problem  $P$  and quota  $k$ ,  $\mathcal{O}^\gamma(P, k) \subseteq IR(P) \cap NW(P)$ .

**Proof** Let  $Q \in \mathcal{E}^\gamma(P, k)$ . It is immediate that  $\gamma(Q) \in IR(P)$ . We prove that  $\gamma(Q) \in NW(P)$ . Suppose to the contrary that  $\gamma(Q) \notin NW(P)$ . Then, there is a student  $i \in I$  and a school  $s \in S$  with  $sP_i\gamma(Q)(i)$  and  $|\gamma(Q)(s)| < q_s$ . Let  $\bar{Q}_i$  be the empty list. Let  $\bar{Q} := (\bar{Q}_i, Q_{-i})$ . By a result of Gale and Sotomayor (1985b, Theorem 2) extended to the college admissions model (Roth and Sotomayor, 1990, Theorem 5.34), for each  $j \in I \setminus i$ , either  $\gamma(\bar{Q})(j) = \gamma(Q)(j)$  or  $\gamma(\bar{Q})(j)Q_j\gamma(Q)(j)$ . Hence, the set of schools to which each  $j \in I \setminus i$  proposes in  $DA(\bar{Q})$  is a subset of the schools to which he proposes in  $DA(Q)$ . Since moreover  $\bar{Q}_i$  is the empty list, each school receives in  $DA(\bar{Q})$  only a subset of the proposals of  $DA(Q)$ . For school  $s$  this immediately implies that  $|\gamma(\bar{Q})(s)| \leq |\gamma(Q)(s)| < q_s$ . So, if we take  $Q'_i = s$  then  $\gamma(Q'_i, Q_{-i})(i) = s$ . Since  $sP_i\gamma(Q)(i)$ ,  $Q'_i$  is a profitable deviation for  $i$  at  $Q$  in  $\Gamma^\gamma(P, k)$ . So,  $Q \notin \mathcal{E}^\gamma(P, k)$ , a contradiction. Hence,  $\gamma(Q) \in NW(P)$ . ■

**Proof of Theorem 6.5** It suffices to prove the proposition for  $k' = k + 1$ . Let  $Q \in \mathcal{E}^\gamma(P, k)$  and suppose that  $Q \notin \mathcal{E}^\gamma(P, k + 1)$ . Hence, there is a student  $i$  and a strategy  $Q'_i \in \mathcal{Q}(k + 1)$  such that  $\gamma(Q'_i, Q_{-i})P_i\gamma(Q'_i, Q_{-i})$ . By Lemma A.2,  $\gamma(Q) \in IR(P)$ . Hence,  $\gamma(Q'_i, Q_{-i})(i) \in S$ . Note also that  $Q'_i$  must be a list containing exactly  $k + 1$  schools, for otherwise it would also be a profitable deviation in  $\Gamma^\gamma(P, k)$ , contradicting  $Q \in \mathcal{E}^\gamma(P, k)$ .

Let  $s$  be the last school listed in  $Q'_i$ . We claim that  $\gamma(Q'_i, Q_{-i})(i) = s$ . Suppose not. Consider the truncation of  $Q'_i$  after  $\gamma(Q'_i, Q_{-i})(i)$  and denote this list by  $Q''_i$ . In other words,  $Q''_i$  is the list obtained from  $Q'_i$  by making all schools listed after  $\gamma(Q'_i, Q_{-i})(i)$  unacceptable. By assumption,  $Q''_i \in \mathcal{Q}(k)$ . It follows from the DA algorithm that

$\gamma(Q''_i, Q_{-i}) = \gamma(Q'_i, Q_{-i})$ . Hence,  $Q''_i$  is a profitable deviation for  $i$  at  $Q$  in  $\Gamma^\gamma(P, k)$ , a contradiction. So,  $\gamma(Q'_i, Q_{-i})(i) = s$ .

Let  $\widehat{Q}_i := s$ . Note  $\widehat{Q}_i \in \mathcal{Q}(k)$ . By Lemma A.1,  $\gamma(\widehat{Q}_i, Q_{-i})(i) = s$ . Hence,  $\widehat{Q}_i$  is a profitable deviation for  $i$  at  $Q$  in  $\Gamma^\gamma(P, k)$ , a contradiction. Hence,  $Q \in \mathcal{E}^\gamma(P, k + 1)$ . ■

We need the following three lemmas to prove Theorem 6.7.

**Lemma A.3** *Let  $f$  be an Ergin-cyclic priority structure. Let  $2 \leq k \leq m$ . Then, there is a school choice problem  $P$  with an unstable equilibrium outcome in the game  $\Gamma^\gamma(P, k)$ , i.e., for some  $Q \in \mathcal{E}^\gamma(P, k)$ ,  $\gamma(Q) \notin S(P)$ .*

**Proof** Since  $f$  is Ergin-cyclic, we may assume, without loss of generality, that schools  $s_1$  and  $s_2$  and students  $i_1, i_2$ , and  $i_3$  constitute an Ergin-cycle. In fact, we may assume, without loss of generality, that

- (a)  $f_{s_1}(i_1) < f_{s_1}(i_2) < f_{s_1}(i_3)$  and  $f_{s_2}(i_3) < f_{s_2}(i_1)$ ,
- (b)  $f_{s_1}(i_j) < f_{s_1}(i_2)$  for  $j \in \{4, \dots, q_{s_1} + 2\}$ , and
- (c)  $f_{s_2}(i_j) < f_{s_2}(i_1)$  for  $j \in \{q_{s_1} + 3, \dots, q_{s_1} + q_{s_2} + 1\}$ .

Consider students' preferences  $P$  given below. (Unacceptable schools are not depicted.)

$P_{i_1}$	$P_{i_2}$	$P_{i_3}$	$P_{i_4}$	$\dots$	$P_{i_{q_{s_1}+2}}$	$P_{i_{q_{s_1}+3}}$	$\dots$	$P_{i_{q_{s_1}+q_{s_2}+1}}$	$P_{i_{q_{s_1}+q_{s_2}+2}}$	$\dots$	$P_{i_n}$
$s_2$	$s_1$	$s_1$	$s_1$	$s_1$	$s_1$	$s_2$	$s_2$	$s_2$			
$s_1$		$s_2$									

We distinguish among three cases for the priority ordering  $f_{s_2}$  of school  $s_2$  with respect to students  $i_1, i_2$ , and  $i_3$ :

- (i)  $f_{s_2}(i_2) < f_{s_2}(i_3) < f_{s_2}(i_1)$ ,
- (ii)  $f_{s_2}(i_3) < f_{s_2}(i_2) < f_{s_2}(i_1)$ , or
- (iii)  $f_{s_2}(i_3) < f_{s_2}(i_1) < f_{s_2}(i_2)$ .

Consider  $DA(P)$ . First note that by construction of  $P$  and (b) and (c), all students in  $\{i_4, \dots, i_{q_{s_1}+q_{s_2}+1}\}$  are assigned a seat at their most preferred school. Since for each  $j \in \{q_{s_1} + q_{s_2} + 2, \dots, n\}$ , student  $i_j$  finds all schools unacceptable, one seat of each of the schools  $s_1$  and  $s_2$  remains to be assigned to the students in  $\{i_1, i_2, i_3\}$ . One easily verifies that in each of the cases (i), (ii), and (iii), the DA algorithm assigns students  $i_1$  and  $i_3$  to schools  $s_1$  and  $s_2$ , respectively. Finally, one readily verifies that there is no other stable matching for  $P$ . Hence, the unique stable matching for  $P$  is  $\mu^* = \gamma(P)$  in which students  $i_1$  and  $i_3$  are assigned a seat at schools  $s_1$  and  $s_2$ , respectively (and student  $i_2$  remains unassigned).

Consider  $Q \in \mathcal{Q}(k)^I$  given below. We will show that  $\gamma(Q) \notin S(P)$ , yet  $Q \in \mathcal{E}^\gamma(P, k)$ .

$Q_{i_1}$	$Q_{i_2}$	$Q_{i_3}$	$Q_{i_4}$	$\cdots$	$Q_{i_{q_{s_1}+2}}$	$Q_{i_{q_{s_1}+3}}$	$\cdots$	$Q_{i_{q_{s_1}+q_{s_2}+1}}$	$Q_{i_{q_{s_1}+q_{s_2}+2}}$	$\cdots$	$Q_{i_n}$
$s_2$		$s_1$	$s_1$	$s_1$	$s_1$	$s_2$	$s_2$	$s_2$			
$s_1$		$s_2$									

Consider  $DA(Q)$ . As before, all students in  $\{i_4, \dots, i_{q_{s_1}+q_{s_2}+1}\}$  are assigned a seat at their most preferred school. One seat of each of the schools  $s_1$  and  $s_2$  remains to be assigned to the students in  $\{i_1, i_2, i_3\}$ . Since  $Q_{i_2}$  is the empty list, and students  $i_1$  and  $i_3$  have different favorite schools at  $Q$ , the DA algorithm assigns in each of the three cases (i), (ii), and (iii), students  $i_1$  and  $i_3$  to schools  $s_2$  and  $s_1$ , respectively. So,  $\gamma(Q) \neq \mu^*$ . Since  $S(P) = \{\mu^*\}$ ,  $\gamma(Q) \notin S(P)$ . Finally, we check that  $Q \in \mathcal{E}^\gamma(P, k)$ . Note that at  $\gamma(Q)$  each of the students  $i_1$  and  $i_3$  is assigned a seat at his favorite school. So, neither student  $i_1$  nor  $i_3$  has a profitable deviation from his strategy  $Q_{i_1}$  and  $Q_{i_3}$ , respectively. One easily verifies that in each of the cases (i), (ii), and (iii), and for each strategy  $Q'_{i_2} \in \mathcal{Q}(k)$ ,  $\gamma(Q)R_{i_2}\gamma(Q_{i_1}, Q'_{i_2}, Q_{i_3})$ . Hence,  $Q \in \mathcal{E}^\gamma(P, k)$ . ■

A mechanism is non bossy if no student can maintain his allotment and cause a change in the other students' allotments by reporting different preferences.

**Definition A.4 Non Bossy Mechanism** (Satterthwaite and Sonnenschein, 1981)

A mechanism  $\varphi$  is *non bossy* if for all  $i \in I$ ,  $Q_i, Q'_i \in \mathcal{Q}$ , and  $Q_{-i} \in \mathcal{Q}^{I \setminus i}$ ,  $\varphi(Q'_i, Q_{-i})(i) = \varphi(Q_i, Q_{-i})(i)$  implies  $\varphi(Q'_i, Q_{-i}) = \varphi(Q_i, Q_{-i})$ .  $\triangle$

**Lemma A.5** *Let  $f$  be an Ergin-acyclic priority structure. Then,  $\gamma$  is non bossy.*

**Proof** Follows from Ergin's (2002) Theorem 1, (iv)  $\rightarrow$  (iii) and proof of (iii)  $\rightarrow$  (ii). ■

**Lemma A.6** *Let  $f$  be an Ergin-acyclic priority structure. Let  $2 \leq k \leq m$ . Then, for any school choice problem  $P$  all equilibrium outcomes in the game  $\Gamma^\gamma(P, k)$  are stable, i.e., for all  $Q \in \mathcal{E}^\gamma(P, k)$ ,  $\gamma(Q) \in S(P)$ .*

**Proof** Suppose to the contrary that  $Q \in \mathcal{E}^\gamma(P, k)$  but  $\gamma(Q) \notin S(P)$ . So, by Lemma A.2, there are two students  $i, j \in I$ ,  $i \neq j$  and a school  $s \in S$  such that  $\gamma(Q)(j) = s$ ,  $sP_i\gamma(Q)(i)$ , and  $f_s(i) < f_s(j)$ .

Since  $\gamma$  is strategy-proof in the unconstrained setting (i.e., when the quota equals  $m$ , the number of schools),  $\gamma(P_i, Q_{-i})R_i\gamma(Q_i, Q_{-i})$ . Let  $Q'_i := \gamma(P_i, Q_{-i})(i)$ . Clearly,  $Q'_i \in \mathcal{Q}(k)$ . By Lemma A.1,  $\gamma(Q'_i, Q_{-i})(i) = \gamma(P_i, Q_{-i})(i)$ . Hence,  $\gamma(Q'_i, Q_{-i})R_i\gamma(Q_i, Q_{-i})$ . If  $\gamma(Q'_i, Q_{-i})P_i\gamma(Q_i, Q_{-i})$ , then  $Q \notin \mathcal{E}^\gamma(P, k)$ , a contradiction. Hence,  $\gamma(Q'_i, Q_{-i})(i) = \gamma(Q_i, Q_{-i})(i)$ .

By Lemma A.5,  $\gamma$  is non bossy. Hence,  $\gamma(P_i, Q_{-i}) = \gamma(Q'_i, Q_{-i}) = \gamma(Q)$ . In particular,  $\gamma(P_i, Q_{-i})(j) = \gamma(Q)(j) = s$ . Since  $sP_i\gamma(Q)(i) = \gamma(P_i, Q_{-i})(i)$ , student  $i$  has justified envy at  $\gamma(P_i, Q_{-i})$ , contradicting  $\gamma(P_i, Q_{-i}) \in S(P_i, Q_{-i})$ . Hence,  $\gamma(Q) \in S(P)$ . ■

**Proof of Theorem 6.7** Follows from Proposition 6.2 and Lemmas A.3 and A.6. ■

## B Appendix: Top Trading Cycles Mechanism, Proofs

We first introduce the following graph-theoretic notation to provide concise proofs of our results. Let  $Q \in \mathcal{Q}^I$ . Suppose the TTC algorithm is applied to  $Q$ , which we will denote by  $TTC(Q)$ , and suppose it terminates in no less than  $l$  steps. We denote by  $G(Q, l)$  the (directed) graph that corresponds to step  $l$ . In this graph, the set of vertices  $V(Q, l)$  is the set of agents present in step  $l$ . For any  $v \in V(Q, l)$  there is a (unique) directed edge in  $G(Q, l)$  from  $v$  to some  $v' \in V(Q, l)$  (possibly  $v' = v$  if  $v \in I$ ) if agent  $v$  points to agent  $v'$ , which will also be denoted by  $e(Q, l, v) = v'$ .

A path (from  $v_1$  to  $v_p$ ) in  $G(Q, l)$  is an ordered list of agents  $(v_1, v_2, \dots, v_p)$  such that  $v_r \in V(Q, l)$  for all  $r = 1, \dots, p$  and each  $v_r$  points to  $v_{r+1}$  for all  $r = 1, \dots, p - 1$ . A self-cycle ( $i$ ) of a student  $i$  is a degenerate path, *i.e.*,  $i$  points to himself in  $G(Q, l)$ . An agent  $v' \in V(Q, l)$  is a follower of an agent  $v \in V(Q, l)$  if there is a path from  $v$  to  $v'$  in  $G(Q, l)$ . The set of followers of  $v$  is denoted by  $F(Q, l, v)$ . An agent  $v' \in V(Q, l)$  is a predecessor of an agent  $v \in V(Q, l)$  if there is a path from  $v'$  to  $v$  in  $G(Q, l)$ . The set of predecessors of  $v$  is denoted by  $P(Q, l, v)$ . A cycle in  $G(Q, l)$  is a path  $(v_1, v_2, \dots, v_p)$  such that also  $v_p$  points to  $v_1$ . Note that a self-cycle is a special case of a cycle. With a slight abuse of notation we sometimes refer to a cycle as the corresponding non ordered set of involved agents. Finally, for  $v \in I \cup S$ , let  $\sigma(Q, v)$  denote the step of the TTC algorithm at which agent  $v$  is removed.

The following observation on the TTC algorithm is key for the results in Section 7.

**Observation B.1** In the TTC algorithm, once a student points to a school it will keep on pointing to the school in subsequent steps until he is assigned to a seat at the school or until the school has no longer available seats. In other words, if  $i \in V(Q, l) \cap I$  for some step  $l$  of  $TTC(Q)$  and  $e(Q, l, i) = s \in S$ , then  $e(Q, r, i) = s$  for all steps  $r$  with  $l \leq r \leq \min\{\sigma(Q, i), \sigma(Q, s)\}$ . Similarly, once a school points to a student it will keep on pointing to the student in subsequent steps until the student is assigned to a seat at this or some other school. In other words, if  $s \in V(Q, l) \cap S$  for some step  $l$  of  $TTC(Q)$  and  $e(Q, l, s) = i \in I$ , then  $e(Q, r, s) = i$  for all steps  $r$  with  $l \leq r \leq \sigma(Q, i)$ .

We now proceed to establish some preliminary results and slightly technical lemmas to be able to prove Theorems 7.1 and 7.2.

**Lemma B.2** *For any school choice problem  $P$  and any quota  $k$ ,  $\mathcal{O}^\tau(P, k) \subseteq IR(P)$ .*

**Proof** Immediate. ■

In order to avoid possible confusion we will sometimes use an additional superindex  $Q$  and write  $q_s^{Q,r}$  instead of  $q_s^r$ .

**Lemma B.3** *Let  $Q \in \mathcal{Q}^I$ . Let  $i \in I$  and  $Q'_i \in \mathcal{Q}$ . Define  $Q' := (Q'_i, Q_{-i})$ . Suppose  $\tau(Q)(i) \neq \tau(Q')(i)$ . Let  $p := \sigma(Q, i)$ ,  $p' := \sigma(Q', i)$ , and  $r := \min\{p, p'\}$ . Then,*

- (a) *at steps  $1, \dots, r-1$ , the same cycles form in  $TTC(Q)$  and  $TTC(Q')$ ;*
- (b)  *$i \in V(Q, r) = V(Q', r)$  and for each school  $s \in V(Q, r) \cap S$ ,  $q_s^{Q,r} = q_s^{Q',r}$ ;*
- (c)  *$e(Q, r, v) = e(Q', r, v)$  for each agent  $v \in V(Q, r) \setminus i$ ;*
- (d) *there is a cycle  $C$  with  $i \in C$  in either  $G(Q, r)$  or  $G(Q', r)$  (but not both).<sup>21</sup>*

**Proof** Item (a) follows from the proof of a result in Abdulkadiroğlu and Sönmez (1999, Lemma 1) or, alternatively, Abdulkadiroğlu and Sönmez (2003, Lemma). As for Item (b), from the definition of  $r$ ,  $i \in V(Q, r) \cap V(Q', r)$ . The remainder of Item (b) follows directly from Item (a). Item (c) follows from Item (b) and the fact that  $Q'_j = Q_j$  for all students  $j \in I \setminus i$ . As for Item (d), by definition of  $r$ , there is a cycle  $C$  with  $i \in C$  in  $G(Q, r)$  or  $G(Q', r)$ . From Item (c) and  $\tau(Q)(i) \neq \tau(Q')(i)$ ,  $e(Q, r, i) \neq e(Q', r, i)$ . In particular,  $C$  is not a cycle in both  $G(Q, r)$  and  $G(Q', r)$ . This proves Item (d). ■

The following definition introduces a class of mechanisms that induce nested Nash equilibria.

#### Definition B.4 Individually Idempotent Mechanism

A mechanism  $\varphi$  is *individually idempotent* if for any  $Q \in \mathcal{Q}^I$ , any  $i \in I$ ,  $Q'_i = \varphi(Q)(i) \in \mathcal{Q}(1)$  implies  $\varphi(Q'_i, Q_{-i}) = \varphi(Q)$ . △

Note that Example 6.3 shows that the Student-Optimal Stable mechanism is *not* individually idempotent:  $\gamma(Q_{i_1}, Q^c, Q_{i_3}) = \{\{i_1, s_1\}, \{i_3, s_2\}, \{i_2\}\}$  but  $\gamma(Q_{i_1}, Q^e, Q_{i_3}) = \{\{i_1, s_2\}, \{i_3, s_1\}, \{i_2\}\}$ . However, as we will see, the Top Trading Cycles mechanism is individually idempotent.

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<sup>21</sup>Note that it is still possible that there is *another* cycle  $\bar{C}$  (i.e.,  $\bar{C} \neq C$ ) with  $i \in \bar{C}$  present in the other graph.

**Proposition B.5** *Let  $\varphi$  be an individually idempotent mechanism. For any school choice problem  $P$  and quotas  $k < k'$ ,  $\mathcal{E}^\varphi(P, k) \subseteq \mathcal{E}^\varphi(P, k')$ .*

**Proof** Let  $Q \in \mathcal{E}^\varphi(P, k)$ . Suppose to the contrary that  $Q \notin \mathcal{E}^\varphi(P, k')$ . Then there exist a student  $i$  and a list  $\bar{Q}_i \in \mathcal{Q}(k')$  such that  $\varphi(\bar{Q}_i, Q_{-i})P_i\varphi(Q)$ . Let  $\bar{Q}'_i := \varphi(\bar{Q}_i, Q_{-i})(i)$ . Clearly,  $\bar{Q}'_i \in \mathcal{Q}(k)$ . Since  $\varphi$  is individually idempotent,  $\varphi(\bar{Q}'_i, Q_{-i}) = \varphi(\bar{Q}_i, Q_{-i})$ . So,  $\varphi(\bar{Q}'_i, Q_{-i})P_i\varphi(Q)$ , contradicting  $Q \in \mathcal{E}^\varphi(P, k)$ . Hence,  $Q \in \mathcal{E}^\varphi(P, k')$ . ■

**Lemma B.6** *Mechanism  $\tau$  is individually idempotent.*

**Proof** Let  $Q \in \mathcal{Q}^I$ . Let  $i \in I$  and define  $Q'_i := \tau(Q)(i) \in \mathcal{Q}(1)$ . Define  $Q' := (Q'_i, Q_{-i})$ . We have to show that  $\tau(Q') = \tau(Q)$ . By non bossiness of  $\tau$ ,<sup>22</sup> it is sufficient to show that  $\tau(Q')(i) = \tau(Q)(i)$ . If  $\tau(Q)(i) = i$ , then from the definition of the TTC algorithm,  $\tau(Q')(i) = i = \tau(Q)(i)$ .

So, suppose  $\tau(Q)(i) =: s \in S$ . Suppose to the contrary that  $\tau(Q')(i) \neq \tau(Q)(i)$ . Then, since  $Q'_i = \tau(Q)(i) = s$ , student  $i$  remains unassigned under  $Q'$ , *i.e.*,  $\tau(Q')(i) = i$ . Let  $p := \sigma(Q, i)$ ,  $p' := \sigma(Q', i)$ , and  $r := \min\{p, p'\}$ . By Lemma B.3(d), there is a cycle  $C$  with  $i \in C$  in either  $G(Q, r)$  or  $G(Q', r)$  (but not both).

Suppose cycle  $C$  is in  $G(Q, r)$  but not in  $G(Q', r)$ . Since student  $i$  is assigned through cycle  $C$  and  $\tau(Q)(i) = s$ ,  $e(Q, r, i) = s$ . Since  $e(Q', r, i) \neq e(Q, r, i)$  and  $Q'_i = \tau(Q)(i) = s$ ,  $e(Q', r, i) = i$ . Hence, at the beginning of step  $r$  of  $TTC(Q')$ , school  $s$  has no available seats, *i.e.*,  $q_s^{Q', r} = 0$ . By Lemma B.3(b),  $q_s^{Q, r} = q_s^{Q', r} = 0$ . So,  $e(Q, r, i) \neq s$ , a contradiction.

So, cycle  $C$  is in  $G(Q', r)$  but not in  $G(Q, r)$ . If  $e(Q', r, i) = s$ , then  $\tau(Q')(i) = s$ , a contradiction with  $\tau(Q')(i) \neq \tau(Q)(i)$ . So by  $Q'_i = \tau(Q)(i) = s$ ,  $e(Q', r, i) = i$ , *i.e.*,  $C = (i)$  is a self-cycle. Since  $i \in V(Q, r)$  and  $\tau(Q)(i) = s$ ,  $q_s^{Q, r} > 0$ . By Lemma B.3(b),  $q_s^{Q', r} = q_s^{Q, r} > 0$ . So,  $s \in V(Q', r)$ . But then from  $Q'_i = s$ ,  $e(Q', r, i) = s$ , a contradiction. We conclude that  $\tau(Q')(i) = \tau(Q)(i)$ . ■

**Proposition B.7** *For any school choice problem  $P$  and quotas  $k < k'$ ,  $\mathcal{E}^\tau(P, k) \subseteq \mathcal{E}^\tau(P, k')$ .*

**Proof** Follows from Proposition B.5 and Lemma B.6. ■

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<sup>22</sup>Pápai's (2000) main result implies that  $\tau$  is group strategy-proof. Group strategy-proofness implies non bossiness.

**Lemma B.8** *Let  $\bar{Q} \in \mathcal{Q}^I$ . Let  $v, v' \in I \cup S$ ,  $v \neq v'$ . Suppose  $v' \in P(\bar{Q}, l, v)$  at some step  $l$  of  $TTC(\bar{Q})$ . Then,  $\sigma(\bar{Q}, v) \leq \sigma(\bar{Q}, v')$  and  $[\sigma(\bar{Q}, v) = \sigma(\bar{Q}, v')$  only if  $v$  and  $v'$  are removed in the same cycle].*

**Proof** By Observation B.1, each agent in the path from  $v'$  to  $v$  will keep on pointing to its follower at least until the step in which agent  $v$  is removed, *i.e.*, step  $\sigma(\bar{Q}, v)$ . Hence,  $\sigma(\bar{Q}, v) \leq \sigma(\bar{Q}, v')$ . Suppose  $\sigma(\bar{Q}, v) = \sigma(\bar{Q}, v')$ . Then, all agents in the path from  $v'$  to  $v$  form part of a cycle at this step. Since an agent can be part of at most one cycle at a given step, all agents in the path from  $v'$  to  $v$  are in the same cycle. ■

**Lemma B.9** *Let  $Q \in \mathcal{Q}^I$ . Let  $i \in I$  and  $Q'_i \in \mathcal{Q}$ . Define  $Q' := (Q'_i, Q_{-i})$ . Suppose  $\tau(Q)(i) \neq \tau(Q')(i)$  and  $\sigma(Q, i) \leq \sigma(Q', i)$ . Then, for each step  $l$  with  $\sigma(Q, i) \leq l \leq \sigma(Q', i)$ , if  $v \in V(Q', l) \setminus (P(Q', l, i) \cup i)$  then  $v \in V(Q, l)$  and  $F(Q, l, v) = F(Q', l, v)$ .*

**Proof** Let  $p := \sigma(Q, i)$  and  $p' := \sigma(Q', i)$ . From Lemma B.3(b),

$$V(Q, p) = V(Q', p) \text{ and } q_s^{Q,p} = q_s^{Q',p} \text{ for each school } s \in V(Q, p) \cap S. \quad (1)$$

With a slight abuse of notation, for each  $l$ ,  $p \leq l \leq p'$ , denote  $P_l = P(Q', l, i) \cup i$ . From Observation B.1,

$$P_p \subseteq P_{p+1} \subseteq \cdots \subseteq P_{p'-1} \subseteq P_{p'}. \quad (2)$$

Also note

$$V(Q', p') \subseteq V(Q', p' - 1) \subseteq \cdots \subseteq V(Q', p + 1) \subseteq V(Q', p). \quad (3)$$

We are done if we prove the following Claim( $l$ ) for each  $l$ ,  $p \leq l \leq p'$ .

CLAIM( $l$ ): If  $v \in V(Q', l) \setminus P_l$ , then  $v \in V(Q, l)$  and  $e(Q, l, v) = e(Q', l, v)$ .

Indeed, Claim( $l$ ) immediately implies the following Consequence( $l$ ):

CONSEQUENCE( $l$ ): If  $v \in V(Q', l) \setminus P_l$ , then  $v \in V(Q, l)$  and  $F(Q, l, v) = F(Q', l, v)$ .

We now prove by induction that Claim( $l$ ) is true for each  $l$ ,  $p \leq l \leq p'$ . By Lemma B.3 (b) and (c),  $V(Q, p) = V(Q', p)$  and  $e(Q, p, v) = e(Q', p, v)$  for each agent  $v \in V(Q, p) \setminus i$ . Hence, Claim( $p$ ) is true.

If  $p' = p$  we are done. So, suppose  $p' \neq p$ . Let  $l$  be a step such that  $p < l \leq p'$ . Assume Claim( $g$ ) is true for all  $g$ ,  $p \leq g < l \leq p'$ . We prove that Claim( $l$ ) is true. Let  $v \in V(Q', l) \setminus P_l$ . From (2) and (3),  $v \in V(Q', g) \setminus P_g$  for each step  $g$ ,  $p \leq g < l$ . From Consequence( $g$ ) ( $p \leq g < l$ ),  $v \in V(Q, g)$  and

$$F(Q, g, v) = F(Q', g, v) \text{ for each step } g, p \leq g < l. \quad (4)$$

Since  $v \in V(Q', l)$ ,  $v$  is not removed at the end of step  $l - 1$  in  $TTC(Q')$ . Then by (1) and (4),  $v$  is also not removed at the end of step  $l - 1$  in  $TTC(Q)$ . Hence,  $v \in V(Q, l)$ .

Assume Claim( $l$ ) is not true, *i.e.*,  $e(Q, l, v) \neq e(Q', l, v)$ . Let  $x := e(Q, l, v)$  and  $x' := e(Q', l, v)$ . Since  $v \notin P_l$ ,  $x' \notin P_l$ . By (2),  $x' \notin P_{l-1}$ . By (3) and  $x' \in V(Q', l)$ ,  $x' \in V(Q', l - 1)$ . By Consequence( $l - 1$ ),  $x' \in V(Q, l - 1)$ . We distinguish between two cases.

CASE 1: Agent  $x'$  is removed at the end of step  $l - 1$  in  $TTC(Q)$ .

From (2) and (3),  $x' \in V(Q', g) \setminus P_g$  for each step  $g$ ,  $p \leq g < l$ . From Consequence( $g$ ) ( $p \leq g < l$ ),  $x' \in V(Q, g)$  and

$$F(Q, g, x') = F(Q', g, x') \text{ for each step } g, p \leq g < l. \quad (5)$$

By (1), (5), and the fact that  $x'$  is removed at the end of step  $l - 1$  in  $TTC(Q)$ ,  $x'$  is also removed at the end of step  $l - 1$  in  $TTC(Q')$ . Hence,  $x' \notin V(Q', l)$ , a contradiction with  $e(Q', l, v) = x'$ .

CASE 2: Agent  $x'$  is *not* removed at the end of step  $l - 1$  in  $TTC(Q)$ .

Then,  $x' \in V(Q, l)$ . Since  $e(Q, l, v) = x$  and  $x \neq x'$ , we have  $xQ_v x'$  (if  $v$  is a student) or  $f_v(x) < f_v(x')$  (if  $v$  is a school). Since  $v \notin P_l$ ,  $v \neq i$ . Hence, since  $e(Q', l, v) = x'$ ,  $x \notin V(Q', l)$ . So, agent  $x$  was removed at some step  $g^*$ ,  $1 \leq g^* \leq l - 1$ , in  $TTC(Q')$ . In fact, by (1),  $p \leq g^* \leq l - 1$ . Note that no agent in  $P_{p'}$  is removed before the end of step  $p'$  in  $TTC(Q')$ . So,  $x \notin P_{p'}$ . By (2),  $x \notin P_{g^*}$ . Hence,  $x \in V(Q', g^*) \setminus P_{g^*}$ . By an argument similar to that of Case 1,  $x$  is also removed at the end of step  $g^*$  in  $TTC(Q)$ . Hence,  $x \notin V(Q, l)$ , a contradiction with  $e(Q, l, v) = x$ . ■

**Lemma B.10** *Let  $Q \in \mathcal{Q}^I$ . Let  $i \in I$  and  $Q'_i \in \mathcal{Q}$ . Define  $Q' := (Q'_i, Q_{-i})$ . Suppose there is a student  $j \in I \setminus i$  with  $\tau(Q)(j) \neq \tau(Q')(j)$ . Then,*

- (a)  $\sigma(Q, i) \leq \sigma(Q, j)$  and  $[\sigma(Q, i) = \sigma(Q, j)$  only if  $i$  and  $j$  are assigned in the same cycle in  $TTC(Q)$ ], and
- (b)  $\sigma(Q', i) \leq \sigma(Q', j)$  and  $[\sigma(Q', i) = \sigma(Q', j)$  only if  $i$  and  $j$  are assigned in the same cycle in  $TTC(Q')$ ].

**Proof** By non bossiness of  $\tau$ ,  $\tau(Q)(i) \neq \tau(Q')(i)$ . Let  $p := \sigma(Q, i)$  and  $p' := \sigma(Q', i)$ . Assume, without loss of generality,  $p \leq p'$ . Then, by definition of  $p$  and Lemma B.3(d), there is a cycle  $C$  in  $G(Q, p)$  with  $i \in C$  but not present in  $G(Q', p)$ .

We first prove (a). By Lemma B.3(a,b), for each student  $h \in I \setminus i$  with  $\sigma(Q, h) < p$  or  $\sigma(Q', h) < p$ ,  $\tau(Q)(h) = \tau(Q')(h)$ . Let  $r := \sigma(Q, j)$  and  $r' := \sigma(Q', j)$ . Since  $\tau(Q)(j) \neq \tau(Q')(j)$ , we have  $r, r' \geq p$ . So,  $\sigma(Q, i) = p \leq r = \sigma(Q, j)$ . Suppose  $\sigma(Q, i) = \sigma(Q, j)$ .

We have to show that  $j \in C$ . Suppose to the contrary that  $j \notin C$ . Then,  $j \in C^*$  for some cycle  $C^*$ ,  $C^* \neq C$ , in  $G(Q, p)$ . Note  $i \notin C^*$ . By Lemma B.3(b),  $V(Q, p) = V(Q', p)$ . Hence, since  $e(Q, p, v) = e(Q', p, v)$  for each agent  $v \in V(Q, p) \setminus i$ ,  $C^*$  is also a cycle in  $G(Q', p)$ . In particular,  $\tau(Q)(j) = \tau(Q')(j)$ , a contradiction. This completes the proof of (a).

We now prove (b). We distinguish between two cases.

CASE 1:  $j \in P(Q', p, i)$ .

Then, (b) follows directly from Lemma B.8 with  $\bar{Q} = Q'$ ,  $v' = j$ , and  $v = i$ .

CASE 2:  $j \notin P(Q', p, i)$ .

Assume that (b) is not true. In other words, assume that  $\sigma(Q', i) > \sigma(Q', j)$  or  $[\sigma(Q', i) = \sigma(Q', j)$  and  $i$  and  $j$  are assigned in different cycles in  $TTC(Q')$ ]. Then,  $\sigma(Q, i) = p \leq r' = \sigma(Q', j) \leq \sigma(Q', i)$ .

Note that by definition of  $r'$ ,  $j \in V(Q', r')$ . Suppose  $j \in (P(Q', r', i) \cup i)$ . Since  $j \neq i$ ,  $j \in P(Q', r', i)$ . By Lemma B.8,  $\sigma(Q', i) \leq \sigma(Q', j)$  and  $[\sigma(Q', i) = \sigma(Q', j)$  only if  $i$  and  $j$  are removed in the same cycle in  $TTC(Q')$ ]. This contradicts the assumption that (b) is not true. So,  $j \notin (P(Q', r', i) \cup i)$ . In other words,  $v \in V(Q', r') \setminus (P(Q', r', i) \cup i)$ . Hence, by Lemma B.9,  $j \in V(Q, r')$  and  $F(Q, r', j) = F(Q', r', j)$ . Since  $\sigma(Q', j) = r'$ , student  $j$  forms part of a cycle, say  $C'$ , in  $G(Q', r')$ . Hence,  $C' = F(Q', r', j)$ . So, also  $C' = F(Q, r', j)$ . Hence, student  $j$  is assigned to the same school (or himself) in  $TTC(Q)$  and  $TTC(Q')$ , contradicting  $\tau(Q)(j) \neq \tau(Q')(j)$ . This completes the proof of (b). ■

**Lemma B.11** *Let  $P$  be a school choice problem. Let  $2 \leq k \leq m$ . Let  $Q \in \mathcal{E}^\tau(P, k)$ . Define  $\bar{Q}_i := \tau(Q)(i)$  for all  $i \in I$ . Then,  $\bar{Q} \in \mathcal{E}^\tau(P, 1)$  and  $\tau(\bar{Q}) = \tau(Q)$ . In particular,  $\mathcal{O}^\tau(P, k) \subseteq \mathcal{O}^\tau(P, 1)$ .*

**Proof** It is sufficient to prove the following claim:

CLAIM: Let  $P$  be a school choice problem. Let  $2 \leq k \leq m$ ,  $Q \in \mathcal{E}^\tau(P, k)$ , and  $j \in I$ . Let  $\tilde{Q}_j := \tau(Q)(j)$ . Then,  $\tilde{Q} := (\tilde{Q}_j, Q_{-j}) \in \mathcal{E}^\tau(P, k)$ .

Indeed, if the Claim holds true we can pick students one after another and eventually obtain a profile  $\bar{Q} \in \mathcal{E}^\tau(P, k)$  where for all  $j \in I$ ,  $\bar{Q}_j = \tau(Q)(j)$ . By construction,  $\bar{Q} \in \mathcal{Q}(1)^I$ . So,  $\bar{Q} \in \mathcal{E}^\tau(P, 1)$ . By repeated use of Lemma B.6,  $\tau(\bar{Q}) = \tau(Q)$ .

To prove the Claim, suppose to the contrary that  $\tilde{Q} \notin \mathcal{E}^\tau(P, k)$ . So, there is a student  $i$  with a profitable deviation at  $\tilde{Q}$  in  $\Gamma^\tau(P, k)$ . In fact, by Lemma B.6 there is a list  $Q'_i \in \mathcal{Q}(1)$  with

$$\tau(Q'_i, \tilde{Q}_{-i}) P_i \tau(Q_i, \tilde{Q}_{-i}). \quad (6)$$

By Lemma B.6,  $\tau(\tilde{Q}) = \tau(Q)$ . We claim that  $i \neq j$ . Suppose  $i = j$ . Then  $\tilde{Q}_{-i} = \tilde{Q}_{-j} = Q_{-j}$ . Hence, (6) becomes  $\tau(Q'_i, Q_{-j})P_j\tau(Q_j, Q_{-j})$  contradicting  $Q \in \mathcal{E}^\tau(P, k)$ . So,  $i \neq j$ .

Recall  $\tilde{Q} = (Q_i, \tilde{Q}_j, Q_{-ij})$ . Define  $\tilde{Q}' := (Q'_i, \tilde{Q}_j, Q_{-ij})$  and  $Q' := (Q'_i, Q_j, Q_{-ij})$ . We can rewrite (6) as

$$\tau(\tilde{Q}') = \tau(Q'_i, \tilde{Q}_j, Q_{-ij})P_i\tau(Q_i, \tilde{Q}_j, Q_{-ij}) = \tau(\tilde{Q}). \quad (7)$$

From  $Q \in \mathcal{E}^\tau(P, k)$  and Lemma B.2,  $\tau(\tilde{Q}) = \tau(Q) \in IR(P)$ . By (7),  $\tau(\tilde{Q}')(i) \in S$ . Let  $s := \tau(\tilde{Q}')(i)$ . Since  $\tilde{Q}'_i = Q'_i \in \mathcal{Q}(1)$ ,  $Q'_i = s$ .

Suppose  $\tau(Q')(j) = \tau(Q)(j)$ . Recall  $\tilde{Q}_j = \tau(Q)(j)$ . So,  $\tilde{Q}_j = \tau(Q')(j)$ . Hence, Lemma B.6 implies  $\tau(Q'_i, \tilde{Q}_j, Q_{-ij}) = \tau(Q'_i, Q_j, Q_{-ij})$  and  $\tau(Q_i, \tilde{Q}_j, Q_{-ij}) = \tau(Q_i, Q_j, Q_{-ij})$ . Then (7) can be rewritten as  $\tau(Q'_i, Q_j, Q_{-ij})P_i\tau(Q_i, Q_j, Q_{-ij})$ . So,  $Q \notin \mathcal{E}^\tau(P, k)$ , a contradiction. Hence,  $\tau(Q')(j) \neq \tau(Q)(j)$ .

We claim that  $\tau(Q')(i) \neq \tau(\tilde{Q}')(i)$ . To prove this, suppose to the contrary that  $\tau(Q')(i) = \tau(\tilde{Q}')(i)$ . Since  $\tau(\tilde{Q}) = \tau(Q)$ , (7) boils down to  $\tau(Q')P_i\tau(Q)$ , which implies that  $Q \notin \mathcal{E}^\tau(P, k)$ , a contradiction. So,  $\tau(Q')(i) \neq \tau(\tilde{Q}')(i)$ .

Note that for any student  $h \neq i$ ,  $Q'_h = Q_h$ . So, by Lemma B.10,  $\sigma(Q', i) \leq \sigma(Q', j)$ . Note also that for any student  $h \neq j$ ,  $\tilde{Q}'_h = Q'_h$ . So, by Lemma B.10,  $\sigma(Q', j) \leq \sigma(Q', i)$ . So,  $\sigma(Q', i) = \sigma(Q', j)$ . From Lemma B.10 it follows that  $i$  and  $j$  are in the same cycle in  $TTC(Q')$ . So,  $i$  is not in a self-cycle. Hence,  $i$  is assigned to a school in  $TTC(Q')$ . Since  $Q'_i = s$ ,  $\tau(Q')(i) = s$ . By definition,  $s = \tau(\tilde{Q}')(i)$ . So,  $\tau(Q')(i) = \tau(\tilde{Q}')(i)$ , a contradiction. Hence,  $\tilde{Q} \in \mathcal{E}^\tau(P, k)$ , which completes the proof of the Claim. ■

**Proof of Theorem 7.2** Follows from Proposition B.7 and Lemma B.11. ■

**Proof of Theorem 7.1** In the unconstrained setting mechanism  $\tau$  is strategy-proof. Hence,  $P \in \mathcal{E}^\tau(P, m)$ . By Theorem 7.2,  $\tau(P) \in \mathcal{O}^\tau(P, k)$  for any  $1 \leq k \leq m$ . ■

In order to prove Theorem 7.5 we need the following two lemmas.

**Lemma B.12** *Let  $f$  be a Kesten-cyclic priority structure. Let  $1 \leq k \leq m$ . Then, there is a school choice problem  $P$  with an unstable equilibrium outcome in the game  $\Gamma^\tau(P, k)$ , i.e., for some  $Q \in \mathcal{E}^\tau(P, k)$ ,  $\tau(Q) \notin S(P)$ .*

**Proof** By Theorem 1 of Kesten (2006), there is a school choice problem  $P$  such that  $\tau(P)$  is unstable. Since  $\tau$  is strategy-proof,  $P \in \mathcal{E}^\tau(P, m)$ . Hence, by Theorem 7.2,  $\tau(P) \in \mathcal{O}^\tau(P, m) = \mathcal{O}^\tau(P, k)$ . So, there is a profile  $Q \in \mathcal{Q}(k)^I$  such that  $Q \in \mathcal{E}^\tau(P, k)$  and  $\tau(Q) = \tau(P) \notin S(P)$ . ■

**Lemma B.13** *Let  $f$  be a Kesten-acyclic priority structure. Let  $1 \leq k \leq m$ . Then, for any school choice problem  $P$  all equilibrium outcomes in the game  $\Gamma^\tau(P, k)$  are stable, i.e., for all  $Q \in \mathcal{E}^\tau(P, k)$ ,  $\tau(Q) \in S(P)$ . In fact,  $S(P) = \mathcal{O}^\tau(P, k)$ .*

**Proof** By Theorem 1 of Kesten (2006),  $\tau = \gamma$ . Hence,  $\mathcal{O}^\tau(P, k) = \mathcal{O}^\gamma(P, k)$ . By Lemma 1 of Kesten (2006),  $f$  is Ergin-acyclic. So, from Theorem 6.7,  $S(P) = \mathcal{O}^\gamma(P, k) = \mathcal{O}^\tau(P, k)$ . ■

**Proof of Theorem 7.5** Follows from Lemmas B.12 and B.13. ■

## C Appendix: Proofs of Results in Section 8

**Proof of Lemma 8.1** Let  $\varphi := \gamma, \tau$ . The result follows directly from the strategy-proofness of  $\gamma$  (Dubins and Freedman, 1981; Roth, 1982) and  $\tau$  (Abdulkadiroğlu and Sönmez, 2003) by using  $Q'_i$  as student  $i$ 's “true preferences.”  $\varphi(Q'_i, Q_{-i})(i)$  is ranked higher than  $\varphi(Q_i, Q_{-i})(i)$  by  $Q'_i$ , hence  $\varphi(Q'_i, Q_{-i})(i)$  is ranked higher than  $\varphi(Q_i, Q_{-i})(i)$  by  $P_i$ . ■

**Proof of Lemma 8.2** Let  $\varphi := \gamma, \tau$ . We will prove that  $\varphi(P_i^k, Q_{-i})(i) = \varphi(Q)(i)$  for all  $Q_{-i} \in \mathcal{Q}(k)^{\wedge i}$  or  $\varphi(P_i^k, Q'_{-i})P_i\varphi(Q_i, Q'_{-i})$  for some  $Q'_{-i} \in \mathcal{Q}(k)^{\wedge i}$ . (This obviously completes the proof as it implies that no strategy  $k$ -dominates  $P_i^k$ .)

Suppose  $\varphi(P_i^k, Q_{-i})(i) \neq \varphi(Q)(i)$  for some  $Q_{-i} \in \mathcal{Q}(k)^{\wedge i}$ . We have to show that for some  $Q'_{-i} \in \mathcal{Q}(k)^{\wedge i}$ ,  $\varphi(P_i^k, Q'_{-i})P_i\varphi(Q_i, Q'_{-i})$ . Suppose that for some  $\tilde{Q}_{-i} \in \mathcal{Q}^{\wedge i}$ ,  $\varphi(Q_i, \tilde{Q}_{-i})P_i\varphi(P_i^k, \tilde{Q}_{-i})$ . Since  $\varphi(P_i^k, \tilde{Q}_{-i})(i)R_i i$ , we have  $\tilde{s} := \varphi(Q_i, \tilde{Q}_{-i})(i) \in S$ . From Lemma A.1 (for  $\gamma$ ) and Lemma B.6 (for  $\tau$ ),  $\varphi(\tilde{s}, \tilde{Q}_{-i})(i) = \varphi(Q_i, \tilde{Q}_{-i})(i)$ .

Suppose  $\tilde{s}$  is also listed in  $P_i^k$ . Then,

$$\varphi(P_i^k, \tilde{Q}_{-i})(i)R_i\varphi(P_i^k, \tilde{Q}_{-i})(i) = \varphi(\tilde{s}, \tilde{Q}_{-i})(i) = \tilde{s}, \quad (8)$$

where  $P_i^k$  is the preference relation obtained from  $P_i^k$  by putting  $\tilde{s}$  in the first position. The first relation follows from Lemma 8.1. The second relation follows from the fact that the assignment by the DA/TTC algorithm does not change if a student makes more schools acceptable *and* puts them below the school he is assigned to. Clearly, (8) contradicts  $\varphi(Q_i, \tilde{Q}_{-i})P_i\varphi(P_i^k, \tilde{Q}_{-i})$ . Hence,  $\tilde{s}$  is not listed in  $P_i^k$ .

Let  $\tilde{S} := \{s \in S \setminus \tilde{s} : sQ_i\tilde{s}\}$ . The fact that  $\tilde{s}$  is not listed in  $P_i^k$  together with the definition of  $P_i^k$  implies that there is a school  $s \notin \tilde{S}$  listed in  $P_i^k$  with  $sP_i\tilde{s}$ . Let  $s^*$  be the  $P_i$ -best school among the schools  $s \notin \tilde{S}$  listed in  $P_i^k$  with  $sP_i\tilde{s}$ .

Suppose  $\varphi = \gamma$ . Since  $\varphi(Q_i, \tilde{Q}_{-i}) \in NW(Q_i, \tilde{Q}_{-i})$ ,  $|\varphi(Q_i, \tilde{Q}_{-i})(s)| = q_s$  for all  $s \in \tilde{S}$ . Clearly, for all  $s \in \tilde{S}$ ,  $i \notin \varphi(Q_i, \tilde{Q}_{-i})(s)$ . Also, for all  $s, t \in \tilde{S}$  with  $s \neq t$ ,  $\varphi(Q_i, \tilde{Q}_{-i})(s) \cap \varphi(Q_i, \tilde{Q}_{-i})(t) = \emptyset$ . So we can define for  $j \in I \setminus i$ ,

$$Q'_j := \begin{cases} s & \text{if } j \in \varphi(Q_i, \tilde{Q}_{-i})(s) \text{ for some } s \in \tilde{S}, \\ \emptyset & \text{otherwise.} \end{cases}$$

By the assumption that  $\varphi = \gamma$ ,  $\varphi(Q_i, \tilde{Q}_{-i}) \in S(Q_i, \tilde{Q}_{-i})$ . Hence,  $f_s(j) < f_s(i)$  for all  $s \in \tilde{S}$  and all  $j \in \varphi(Q_i, \tilde{Q}_{-i})(s)$ . From  $\varphi(Q_i, Q'_{-i}) \in S(Q_i, Q'_{-i})$  and the definition of  $Q'_{-i}$ ,  $\varphi(Q_i, Q'_{-i})(i) = \tilde{s}$ . Similarly,  $\varphi(P_i^k, Q'_{-i})(i) = s^*$ . By definition of  $s^*$ ,  $\varphi(P_i^k, Q'_{-i})(i) = s^*P_i\tilde{s} = \varphi(Q_i, Q'_{-i})(i)$ , which completes the proof for the case  $\varphi = \gamma$ .

Suppose  $\varphi = \tau$ . For any  $s \in \tilde{S}$  define  $I_s$  as the set of students  $j$  that are assigned a seat through a cycle (in the TTC algorithm) of which school  $s$  is part and such that  $s$  points to  $j$ . Formally,

$$I_s := \left\{ j \in I : e \left( (Q_i, \tilde{Q}_{-i}), \sigma[(Q_i, \tilde{Q}_{-i}), j], s \right) = j \in P \left( (Q_i, \tilde{Q}_{-i}), \sigma[(Q_i, \tilde{Q}_{-i}), j], s \right) \right\}.$$

From Observation B.1 and  $sQ_i\tilde{s} = \varphi(Q_i, \tilde{Q}_{-i})(i)$  for all  $s \in \tilde{S}$  it follows that for all  $s \in \tilde{S}$ ,  $i \notin I_s$  and  $|I_s| = q_s$ . Also, for all  $s, t \in \tilde{S}$  with  $s \neq t$ ,  $I_s \cap I_t = \emptyset$ . So we can define for  $j \in I \setminus i$ ,

$$Q'_j := \begin{cases} s & \text{if } j \in I_s \text{ for some } s \in \tilde{S}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since  $sQ_i\tilde{s} = \varphi(Q_i, \tilde{Q}_{-i})(i)$  for all  $s \in \tilde{S}$  it follows that  $f_s(j) < f_s(i)$  for all  $s \in \tilde{S}$  and all  $j \in I_s$ . From the definition of  $Q'_{-i}$  and the TTC algorithm,  $\varphi(Q_i, Q'_{-i})(i) = \tilde{s}$ . Similarly,  $\varphi(P_i^k, Q'_{-i})(i) = s^*$ , which completes the case  $\varphi = \tau$  and hence the proof. ■

**Proof of Proposition 8.5** By definition of the DA algorithm,  $|M(\gamma(P^k))| \leq |M(\gamma(P))|$ . We complete the proof by showing that if  $i \in M(\gamma(P))$ , then  $\gamma(P^k)R_i\gamma(P)$ . (Since  $\gamma(P) \in IR(P)$ ,  $\gamma(P^k)(i) \in S$ . Hence,  $i \in M(\gamma(P^k))$ . But then  $M(\gamma(P^k)) = M(\gamma(P))$ .)

Let  $i \in M(\gamma(P))$ . Denote  $s := \gamma(P)(i) \in S$ . Suppose to the contrary that  $sP_i\gamma(P^k)(i)$ . Let  $Q'_i := s$ . By Lemma A.1,  $\gamma(Q'_i, P_{-i})(i) = s$ . By a result of Gale and Sotomayor (1985b, Theorem 2) extended to the college admissions model (Roth and Sotomayor, 1990, Theorem 5.34),  $Q'_i$  ranks  $\gamma(Q'_i, P_{-i}^k)(i)$  weakly higher than  $\gamma(Q'_i, P_{-i})(i)$ . So,  $\gamma(Q'_i, P_{-i}^k)(i) = s$ , contradicting the assumption that  $P^k \in \mathcal{E}(P, k)$ . So,  $\gamma(P^k)(i)R_i s = \gamma(P)(i)$ . ■

**Proof of Proposition 8.6** In Example 8.3,  $\gamma(P) = \{\{i_1, s_1\}, \{i_2, s_3\}, \{i_3, s_2\}\}$  and  $\tau(P) = \{\{i_1, s_1\}, \{i_3, s_3\}, \{i_2\}, \{s_2\}\}$ . So,  $|M(\tau(P))| = 2 < 3 = |M(\gamma(P))|$ .

In Example 8.4,  $\gamma(P) = \{\{i_2, s_3\}, \{i_3, s_2\}, \{i_1\}, \{s_1\}\}$  and  $\tau(P) = \{\{i_1, s_2\}, \{i_2, s_3\}, \{i_3, s_1\}\}$ . So,  $|M(\tau(P))| = 3 > 2 = |M(\gamma(P))|$ . ■

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