

# Sharing a River among Satiabile Agents\*

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## Abstract

We consider the problem of efficiently sharing water from a river among a group of satiable agents. Because each agent's benefits from consuming water exhibit a satiation point (or are single-peaked), the problem becomes a cooperative game with externalities. The value of a coalition for a given partition of the agents is determined by the outcome of the backwards induction algorithm of a dynamic perfect information game induced by the partition and the locations of the agents along the river. Depending on the behavior of the agents outside a coalition, the core may be empty or not. If agents outside the coalition do not cooperate (i.e. they form singletons), then the downstream incremental distribution is the unique core distribution which is fair according to the "aspiration welfare" principle. If they all cooperate (i.e. they form a coalition), then the core may be empty. Furthermore, there exists a fair distribution satisfying all core lower bounds for all connected coalitions if and only

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if each agent's individual rationality constraint is independent of the behavior of the other agents.

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## 1 Introduction

In many economic and political environments the characteristics of a prisoner's dilemma are present: the non-cooperative equilibrium is inefficient and enforcing the efficient outcome requires cooperation (Hardin, 1968; Ostrom, 1990). International agreements determine how to achieve cooperation among a group of countries and specify how to make monetary compensations to distribute the resulting benefits. Examples are the European Union, the GATT, and the Kyoto Protocol on greenhouse emissions.

In all these environments the benefit of a group of countries depends on *how* the other countries behave. For instance in the Kyoto Protocol, if the countries outside of the Kyoto protocol continue to pollute as before, then the benefit of the countries, which signed the protocol, is smaller than when the countries outside agreed to reduce greenhouse emissions a protocol similar to the Kyoto one. Therefore, cooperation of the other countries exerts an externality on the value (or benefit) of a coalition (or group) of countries. In other words, in these environments the countries play a cooperative game with externalities.

A natural requirement of any agreement is that a subgroup of countries should not be better off by signing a separate agreement. Otherwise it is credible to threaten the agreement and acquire a separate one which blocks the initial one. An agreement belongs to the core if it is not blocked by a subgroup of countries. In the presence of externalities the stability of an agreement depends on how countries act after the deviation of a coalition. We might make two extreme assumptions about the behavior of countries outside the blocking coalition: either they continue to cooperate by signing their own agreement or they do not cooperate at all. The non-cooperative

core requires that the agreement is not blocked by a subgroup assuming that the countries outside do not cooperate. This concept corresponds to Hart and Kurtz (1983)'s notion of  $\gamma$ -stability of agreements whereby an agreement is disbanded once a coalition deviates. Similarly, the cooperative core imposes that the agreement is not blocked by a subgroup assuming that the countries outside do all cooperate. This concept corresponds to Hart and Kurz (1983)'s notion of  $\delta$ -stability whereby countries continue to act together after the deviation of a coalition.

We consider international agreements for sharing water resources of a river. The importance of this problem has been empirically shown by Godana (1985) and Barrett (1994). The field of research on water allocation is increasingly important with diminishing water reserves (Young and Haveman, 1995; Carraro, Marchiori, and Sgobbi, 2005, Griffin, 2006). Our paper will be the first one to consider water allocation *with externalities*. We follow Kilgour and Dinar (2001) and Ambec and Sprumont (2002) and consider a set of countries which is located along the river. Each country extracts water from the river for consumption and/or production. The river picks up volume along its course. Agents value water differently in the sense that some have higher needs and higher marginal utility /productivity than others. These heterogeneous valuations are represented by concave and single-peaked benefit functions, where the peak consumption corresponds to a country's satiation point. Water is scarce and it is not possible that everybody consumes his peak. Free access extraction of water is inefficient. Typically, countries located upstream consume too much (e.g. up to their satiation points), thereby leaving not enough to supply downstream users. An efficient allocation of water maximizes the total welfare (i.e. the sum of the countries' benefits). Such an allocation may require upstream countries to limit their own consumption in order to increase the consumption of downstream countries whose marginal benefits are higher. Clearly, inducing the upstream countries to do so requires compensatory payments. These payments together with an efficient allocation of water generate a distribution of the total welfare of the grand coalition. We exam-

ine which distributions are acceptable for the countries according to certain fairness criteria.

Kilgour and Dinar (2001) and Ambec and Sprumont (2002) considered the special case when each country's benefit function is strictly increasing and satiation points do not exist. This assumption appears unnatural because in reality overconsumption may cause flooding or increase sanitation costs with higher water extraction costs. The production function and/or utility from water consumption is therefore decreasing after satiation (e.g. Griffin, 2006, Chap 2). We show that under single-peaked benefit functions the countries located along the river play a cooperative game with externalities.<sup>1</sup> The intuition is as follows. Because water is freely disposable, any country never consumes more than his satiation point. Therefore, for a coalition it may be profitable to pass some water from one of its connected components to another although some water is consumed by countries located in between these two components. Since the same is true for the countries outside of a coalition, this implies that the value of a coalition depends on how the other countries cooperate or behave. For any coalition and any partition of the countries (such that the coalition is a member of the partition), the structure of the river naturally defines a dynamic game with perfect information: the players are the members of the partition and the nodes of play are given by the connected components of all members of the partition. Then the backwards induction algorithm calculates the equilibrium water consumption plan. The value of a coalition under the given partition is simply the sum of the benefits at the equilibrium plan of the countries belonging to the coalition.

Because property rights over water are not well defined, there are two conflicting doctrines invoked by riparian countries in international river disputes: the theory of

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<sup>1</sup>Contrary to most of the literature on cooperative game theory, our environment does not rule out that interactions between the other countries exert an externality on a coalition. For cooperative games with externalities several recent papers offer extensions of the Shapley value of games without externalities (see Maskin (2003), de Clippel and Serrano (2005), Navarro (2006) and Macho-Stadler, Pérez-Castrillo, and Wettstein (2006)).

*absolute territorial sovereignty* (ATS) and the theory of *unlimited territorial integrity* (UTI) respectively (see Godona, 1985). Core lower bounds are inspired by ATS. Under UTI a country (or group of countries) could freely use the full stream of water originating upstream from its location if the other countries are absent, thereby enjoying a benefit called “aspiration welfare”. Since water is scarce, not everybody can enjoy its aspiration welfare. A welfare distribution that assigns to any country or group of countries more than its aspiration welfare should be perceived as unfair. The aspiration welfare defines upper bounds on welfare for any coalition of countries.<sup>2</sup>

Under non-cooperative behavior there exist distributions satisfying the core lower bounds. Our first main result shows that the downstream incremental distribution is the unique distribution satisfying the non-cooperative core lower bounds and the aspiration upper bounds. The downstream incremental distribution is the incremental distribution corresponding to the natural order of the river. Our second main result shows that for more than three countries, there may not exist any distribution satisfying the cooperative core lower bounds. Therefore, the cooperative core lower bounds are above the non-cooperative core lower bounds. In general cooperation exerts a positive externality on the value of a coalition compared to its value under non-cooperative behavior.

Our first two main results are consistent with the literature on international agreements for pollution reduction. This literature disagrees on the stability of a global agreement (the “grand coalition”) because of opposite assumptions about the behavior of the nonmembers of an agreement. On the one hand, Chander and Tulkens (1997) show that the non-cooperative is non-empty, thereby leading to a “grand coalition” agreement. On the other hand, Carraro and Siniscalco (1993) assume that coalitions still cooperate when an individual country deviates. They conclude that any global agreement is not stable because at least one individual country blocks it and the core is empty.

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<sup>2</sup>Notice that, in a recent paper, Ni and Wang (2007) apply the ATS and UTI principles to the problem of dividing the cost of cleaning a polluted river.

An important work related to ours is Demange (2004). She considers hierarchies and shows that the “hierarchical outcome” satisfies the core lower bounds for all connected coalitions<sup>3</sup> for all super-additive cooperative games. If we insist that the core lower bounds are satisfied for some non-connected coalitions, then there exists a large class of super-additive games where the “hierarchical outcome” violates the core lower bounds. If the hierarchy is a river, then the hierarchical outcome corresponds to the upstream incremental distribution. Both her and our work have in common that the cooperative game is super-additive and that an incremental distribution corresponding to the structure of the river (or the hierarchy) is proposed as a solution to the game under consideration. The important differences between Demange (2004) and our work are that here externalities do exist whereas in hers they do not and that the downstream incremental distribution satisfies the non-cooperative core lower bounds for all coalitions (connected or non-connected).

Since the core is empty, similarly to Demange (2004) we may allow only connected coalitions to block. Even if blocking is restricted to these coalitions, the core may still be empty. Our third main result gives a simple necessary and sufficient condition for the existence of a distribution satisfying the aspiration upper bounds and the UTI doctrine for all connected coalitions independently of the other countries’ behavior. The condition is that cooperation exerts no externality on the value of any country. Since all core lower bounds are above the non-cooperative core lower bounds, it follows that downstream incremental distribution is not blocked by any connected coalition independently of the other countries’ behavior if and only if the individual rationality constraints are identical under all behaviors of the other countries.

The paper is organized as follows. In Section 2 we introduce the problem of sharing a river among satiable agents (or countries) and we determine necessary and sufficient conditions for an efficient water consumption plan. In Section 3 we calculate the value of a coalition for each partition of the agents via the backwards induction

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<sup>3</sup>A coalition  $S$  is connected if for any two agents belonging to  $S$ , any agent in between those agents also belongs to  $S$ .

algorithm applied to a dynamic game induced by the structure of the river and the partition of the agents. In Section 4 we focus on non-cooperative behavior and show that the downstream incremental distribution is the unique distribution satisfying the non-cooperative core lower bounds and the aspiration upper bounds. In Section 5 we turn to cooperative behavior and show that for more than three agents there may not exist any distribution satisfying the cooperative core lower bounds. Furthermore, the downstream incremental distribution satisfies all core lower bounds for all connected coalitions if and only if the cooperation exerts no externality on the value of any agent.

## 2 The Problem

Let  $N = \{1, \dots, n\}$  denote the set of agents (or countries). We identify agents with their locations along the river and number them from upstream to downstream:  $i < j$  means that  $i$  is upstream from  $j$ . A coalition is a non-empty subset of  $N$ . Given two coalitions  $S$  and  $T$ , we write  $S < T$  if  $i < j$  for all  $i \in S$  and all  $j \in T$ . Given a coalition  $S$ , we denote by  $\min S$  and  $\max S$ , respectively, the smallest and largest members of  $S$ . Let  $Pi = \{1, \dots, i\}$  denote the set of predecessors of agent  $i$  and  $P^0i = Pi \setminus \{i\}$  denote the set of strict predecessors of agent  $i$ . Similarly, let  $Fi = \{i, i + 1, \dots, n\}$  denote the set of followers of agent  $i$  and let  $F^0i = Fi \setminus \{i\}$  denote the set of strict followers of  $i$ . A coalition  $S$  is connected if for all  $i, j \in S$  and all  $k \in N$ ,  $i < k < j$  implies  $k \in S$ . Given a coalition  $S$ , let  $\mathcal{C}(S)$  denote the set of connected components of  $S$ , i.e.  $\mathcal{C}(S)$  is the coarsest partition of  $S$  such that any  $T \in \mathcal{C}(S)$  is connected. We often omit set brackets for sets and write  $i$  instead of  $\{i\}$  or  $v(i, j)$  instead of  $v(\{i, j\})$ .

The river picks up volume along its course. We denote by  $e_i \geq 0$  the volume which the river picks up at agent  $i$ 's location (or in country  $i$ ). Each agent is endowed with a benefit function. Let  $b_i : \mathbb{R}_+ \rightarrow \mathbb{R}$  denote agent  $i$ 's benefit function. We assume that  $b_i$  is differentiable for all  $x_i > 0$  and strictly concave. Furthermore,  $b'_i(x_i)$  goes to

infinity as  $x_i$  tends to 0 and there exists a satiation point  $\hat{x}_i > 0$  such that  $b'_i(\hat{x}_i) = 0$ . In other words,  $\hat{x}_i$  is agent  $i$ 's optimal (water) consumption and if he consumes more than  $\hat{x}_i$ , then he will infer a loss (compared to consuming  $\hat{x}_i$ ) from overconsumption.

A problem is a triple  $(N, e, b)$  where  $e = (e_i)_{i \in N}$  and  $b = (b_i)_{i \in N}$ . Given a problem, a consumption plan for  $N$  is a vector  $x(N) \in \mathbb{R}_+^N$  such that for all  $j \in N$

$$\sum_{i \in P_j} x_i(N) \leq \sum_{i \in P_j} e_i.$$

The above constraint says that the water  $e_i$ , which is picked up by the river at agent  $i$ 's location, can only be consumed by  $i$  and the agents which are located downstream from  $i$ . This makes our problem different from both the allocation of a private good with the possibility of sidepayments and queuing problems where the order of the agents is flexible and agents are compensated for the welfare maximizing queue (see among others Maniquet (2003) and Chun (2004)).

Given a consumption plan  $x(N)$  and an agent  $i$ , let

$$E_i(x(N)) = \sum_{j \in P^0 i} (e_j - x_j(N))$$

denote the amount of water which is passed to agent  $i$  from his strict predecessors  $P^0 i$  in the consumption plan  $x(N)$  (with the convention  $E_1(x(N)) = 0$ )<sup>4</sup>.

We call  $x^*(N)$  an optimal (or efficient) consumption plan if and only if it maximizes the sum of all agents' benefits. Note that here it may be suboptimal to use all the water  $\sum_{i \in N} e_i$ . In particular, it is suboptimal for any agent to consume more than  $\hat{x}_i$ . Now analogously as in Ambec and Sprumont (2002) we can show that there exists a unique optimal consumption plan  $x^*(N)$  (uniqueness follows from the strict concavity of the  $b_i$ ) and that for  $x^*(N)$  there exists a partition  $\{N_k\}_{k=1, \dots, K}$  of  $N$  and

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<sup>4</sup>Agent 1 does not receive any water from the other agents because agent 1 occupies the first location of the river.

a list  $(\beta_k)_{k=1}^K$  of non-negative numbers such that<sup>5</sup>

$$N_k < N_{k'} \text{ and } \beta_k > \beta_{k'} \text{ whenever } k < k' \quad (1)$$

$$b'_i(x_i^*(N)) = \beta_k \text{ for every } i \in N_k \text{ and every } k = 1, \dots, K \quad (2)$$

$$x_i^*(N) \leq \hat{x}_i \text{ for all } i \in N \quad (3)$$

$$\sum_{i \in N_k} (x_i^*(N) - e_i) = 0 \text{ for every } k = 1, \dots, K - 1. \quad (4)$$

Thus, if  $x_i^*(N) = \hat{x}_i$ , then  $i \in N_K$ , i.e. the saturated agents belong to the last member  $N_K$  of the partition.

Notice that in the special case of a “lake”, all river inflow comes from the source  $e_1$  and  $e_i = 0$  for all  $i > 1$ . Then we have  $N_1 = N$  and the agents’ marginal utilities are equalized according to the shadow value of the resource. This defines the efficient allocation of water extracted from a common pool.

More generally, the efficient allocation divides  $N$  into subsets of agents  $N_k$  (or subrivers) who share the “pool”  $\sum_{i \in N_k} e_i$  of water they control among them so as to equalize marginal benefits according to water’s shadow value. Water being more scarce upstream, the shadow value  $\beta_k$  decreases moving downstream from one subset of agents  $N_k$  to another  $N_{k+1}$ .

Money is available in unbounded quantities to perform sidepayments. Agent  $i$ ’s utility from consuming  $x_i$  units of water and the monetary transfer  $t_i$  is  $u_i(x_i, t_i) = b_i(x_i) + t_i$ . An allocation is a tuple  $(x(N), t(N))$  where  $x(N)$  is a consumption plan for  $N$  and  $t(N) \in \mathbb{R}^N$  is a vector of monetary transfers such that  $\sum_{i \in N} t_i(N) \leq 0$ . A (welfare) distribution is any vector  $z = (z_1, \dots, z_n) \in \mathbb{R}^N$  which is the utility image of some allocation  $(x(N), t(N))$  in the sense that  $z_i = b_i(x_i(N)) + t_i(N)$  for all  $i \in N$ . We distribute the maximal welfare  $\sum_{i \in N} b_i(x_i^*(N))$  among the agents.<sup>6</sup>

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<sup>5</sup>For a detailed description of the efficient allocation of water along a river, see Kilgour and Dinar (2001). Furthermore, recall that  $S < T$  means  $i < j$  for all  $i \in S$  and all  $j \in T$ .

<sup>6</sup>Note that any vector  $z \in \mathbb{R}^N$  such that  $\sum_{i \in N} z_i = \sum_{i \in N} b_i(x_i^*(N))$  is a distribution because it is the utility image of  $(x^*(N), t^*(N))$  where  $t_i^*(N) = z_i - b_i(x_i^*(N))$  for all  $i \in N$ .

### 3 Externalities and Core Lower Bounds

Since each agent's benefit function is single-peaked, any agent never consumes more than his satiation point. If marginal benefits are higher for agents located more downstream, then it may be profitable for a coalition to pass some water from one component to another component even though some of the passed water is consumed by agents in between the two components. Therefore, the value of a coalition may be greater than the sum of the values of its connected components. However, it may be also profitable for the agents outside of  $S$  to pass some water from one component to the next one leaving some water for consumption for the agents in  $S$ . Hence, *the value of a coalition  $S$  will depend on both the components of  $S$  and the behavior of the agents outside of  $S$* . In other words, the behavior of the agents outside of  $S$  exerts an externality on the value of coalition  $S$ . In what follows we will assume that the agents outside of  $S$  form a partition and each member of the partition is maximizing its surplus for any amount of water which is not used by the predecessors. Furthermore, by the structure of the river, any amount of unused water can only be transferred downstream and each member of the partition is maximizing its surplus at any of its connected components for any amount of water, which is not used by the predecessors of this connected component. Therefore, the outcome is a "subgame perfect Nash equilibrium of the dynamic game with perfect information given by the river".

Let  $v(S, \mathcal{P})$  denote the value of  $S$  when the partition  $\mathcal{P}$  of  $N$  forms where  $S \in \mathcal{P}$ . The calculation of  $v(S, \mathcal{P})$  follows the simple backwards induction algorithm along the river. Here each coalition belonging to  $\mathcal{P}$  is a player in the extensive form game with perfect information (given by the river). The nodes of play are given by the connected components of all coalitions in  $\mathcal{P}$ . Information is perfect because at any node of play the amount of unused water from the strict predecessors is observed (or equivalently the consumptions of the strict predecessors are observed). A subgame consists of an initial node of play and an amount of unused water which is passed

to the initial node of play by its strict predecessors. In the subgame each node of play, which (weakly) follows the initial one, receives an amount of unused water from its strict predecessors (or equivalently observes the consumption plans chosen by the previous nodes) and chooses a feasible consumption plan given this amount of unused water. The backwards induction algorithm calculates for each subgame the feasible consumption plan of the initial node which maximizes the sum of their benefits plus the sum of the benefits of all components which belong to the same coalition and are further down the river. Here the reactions of the components further down the river are already given by the amount of water which the initial component passes to the following component. The outcome of the backwards induction algorithm is the consumption plan of the (sub)game starting with the first component of the river (agent 1 belongs to this component) and no amount of unused water is received by this first component. Then  $v(S, \mathcal{P})$  is equal to the sum of the benefits all agents belonging to  $S$  receive in the outcome of the backwards algorithm.

Formally, let  $\cup_{T \in \mathcal{P}} \mathcal{C}(T) = \{T_1, \dots, T_k\}$  be such that  $T_1 < \dots < T_k$ . The backwards induction algorithm calculates for each component and each amount of unused water received by this component a feasible consumption plan which is optimal for this component and the components further down the river which belong to the same coalition:<sup>7</sup>

- (k) For all  $E' \geq 0$ , let  $x^*(T_k, E')$  be the optimal consumption plan for  $(T_k, (e_{\min T_k} + E', e_{T_k \setminus \{\min T_k\}}, b_{T_k}))$ ;
- (k-1) For all  $E' \geq 0$ , let  $x^*(T_{k-1}, E')$  be the optimal consumption plan for  $(T_{k-1}, (e_{\min T_{k-1}} + E', e_{T_{k-1} \setminus \{\min T_{k-1}\}}, b_{T_{k-1}}))$ ; note that  $T_{k-1}$  and  $T_k$  necessarily belong to different members of  $\mathcal{P}$ ; after the choice of  $x^*(T_{k-1}, E')$ , the amount  $E_k(x^*(T_{k-1}, E')) = E' + \sum_{i \in T_{k-1}} (e_i - x_i^*(T_{k-1}, E'))$  of unused water is passed from  $T_{k-1}$  to  $T_k$  and  $T_k$  chooses the consumption plan  $x^*(T_k, E_k(x^*(T_{k-1}, E')))$ .
- $\vdots$

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<sup>7</sup>For any  $S \subseteq N$ , let  $b_S = (b_i)_{i \in S}$  and  $e_S = (e_i)_{i \in S}$ .

(1) Given  $E'$  and the volume the river picks up along the locations in  $T_l$ ,  $x(T_l, E')$  is a feasible consumption plan for  $T_l$  if  $E' + \sum_{i \in T_l \cap P_j} (e_i - x_i(T_l, E')) \geq 0$  for all  $j \in T_l$ . By backwards induction, suppose that for all components  $T_{l'}$  following  $T_l$  ( $l' \in \{l+1, \dots, k\}$ ) and all amounts of water  $E' \geq 0$  we have defined  $x^*(T_{l'}, E')$ . Given these choices, a fixed  $E' \geq 0$  and a feasible consumption plan  $x(T_l, E')$ , let  $E_{l+1}(x(T_l, E')) = E' + \sum_{i \in T_l} (e_i - x_i(T_l, E'))$  be the amount of water passed from  $T_l$  to  $T_{l+1}$ , let  $E_{l+2}(x(T_l, E')) = E_{l+2}(x^*(T_{l+1}, E_{l+1}(x(T_l, E'))))$  be the amount of water passed from  $T_l$  and  $T_{l+1}$  to  $T_{l+2}$ , and in general, for  $t \in \{1, \dots, k-l\}$ , let  $E_{l+t}(x(T_l, E')) = E_{l+t}(x^*(T_{l+t-1}, E_{l+t-1}(x(T_l, E'))))$  be the amount of water passed from  $T_l, \dots, T_{l+t-1}$  to  $T_{l+t}$ .

Let  $T \in \mathcal{P}$  be such that  $T_l \subseteq T$ . Then for all  $E' \geq 0$ , let  $x^*(T_l, E')$  be the consumption plan for  $T_l$  which solves

$$\max_{x(T_l, E')} \sum_{i \in T_l} b_i(x_i(T_l, E')) + \sum_{\nu \in \{l+1, \dots, k\}: T_{\nu} \subseteq T} \sum_{i \in T_{\nu}} b_i(x_i^*(T_{\nu}, E_{\nu}(x(T_l, E'))))$$

where  $x(T_l, E')$  is a feasible consumption plan for  $T_l$  given  $E'$ . In other words,  $x(T_l, E')$  maximizes the surplus of  $T$  in the subgame starting at  $T_l$  given  $E'$  and how the other components react on any amount of water which arrives at each component following  $T_l$ .

⋮

From the concavity of the  $b_i$  we obtain that each component's optimal consumption plan is unique. We denote the outcome of the backwards induction algorithm applied to  $\mathcal{P}$  by  $x^{\mathcal{P}}(N)$  where  $x_{T_1}^{\mathcal{P}}(N) = x^*(T_1, 0)$  and for all  $l \in \{2, \dots, k\}$ ,  $x_{T_l}^{\mathcal{P}}(N) = x^*(T_l, E_l(x^*(T_1, 0)))$ . Then for  $S \in \mathcal{P}$  we define

$$v(S, \mathcal{P}) = \sum_{i \in S} b_i(x_i^{\mathcal{P}}(N)).$$

We will also call  $v(S, \mathcal{P})$  the core lower bound of  $S$  given that partition  $\mathcal{P}$  of  $N$  forms.

**Remark 1** The outcome of the backwards induction algorithm may not be unique because some coalitions may be indifferent between passing water and not passing any

water. For example, if there are three agents, then coalition  $\{1, 3\}$  may be indifferent between passing some water from 1 to 3 (and losing some water to agent 2) and not passing any water (and agents 1 and 3, respectively, consume  $e_1$  and  $e_3$ ). In the rare case of indifference at the outcome  $x^{\mathcal{P}}(N)$ , we assume that any coalition is passing water instead of not passing any water. Given  $\mathcal{P}$ , this assumption ensures that the value of any coalition  $S \in \mathcal{P}$  is maximal among all outcomes of the backwards induction algorithm. Because we do not know which subgame perfect equilibrium will arise from  $\mathcal{P}$ , it is sensible to require that the core lower bounds are met for all outcomes of the backward induction algorithm. By the above fact, this is equivalent to the core lower bounds for the outcome of the backwards induction algorithm where in the case of indifference under  $x^{\mathcal{P}}(N)$  water is passed.

Of course, it is a Nash equilibrium where any connected component consumes any amount of passed water. At the outcome of this equilibrium no water is passed between any two connected components belonging to the same coalition. However, this equilibrium is not credible because by the structure of the river water is passed only downstream and the connected components, which are located more downstream, will not consume more than their satiation points. Therefore, we need to focus on subgame perfect Nash equilibrium.

**Remark 2** The following is an important observation. Suppose that for some agent we have  $e_i > \hat{x}_i$ . Then for any partition  $\mathcal{P}$  of  $N$ , in the outcome of the backwards induction algorithm agent  $i$  will never consume more than  $\hat{x}_i$ , i.e.  $x_i^{\mathcal{P}}(N) \leq \hat{x}_i$  and  $i$  will always dispose  $e_i - \hat{x}_i$ . Now define  $e'_i = \hat{x}_i$  and  $e'_{i+1} = e_{i+1} + (e_i - \hat{x}_i)$  and let  $e' = (e_{N \setminus \{i, i+1\}}, e'_i, e'_{i+1})$ .<sup>8</sup> It is immediate that for both problems  $(N, e, b)$  and  $(N, e', b)$ ,  $x^{\mathcal{P}}(N)$  is the outcome of the backwards induction algorithm applied to  $\mathcal{P}$ . Furthermore, we also obtain from (1)-(4) that  $x^*(N)$  is an optimal consumption plan for the problem  $(N, e, b)$  if and only if  $x^*(N)$  is an optimal consumption plan for the

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<sup>8</sup>If  $i = n$ , then we set  $e' = (e_{N \setminus \{n\}}, e'_n)$ , i.e. the amount  $e_n - \hat{x}_n$  is not consumed by any agent.

problem  $(N, e', b)$ . Thus, from now on we may suppose without loss of generality that in the problem  $(N, e, b)$  we have  $e_i \leq \hat{x}_i$  for all  $i \in N$ .

**Remark 3** In the special case of a “lake” ( $e_i = 0$  for every  $i > 1$ ), typically the water flow from the source exceeds the first agent’s satiation level, i.e.  $e_1 > \hat{x}_1$ . In this problem agent 1 consumes at most  $\hat{x}_1$ , passing the rest of water  $e_1 - \hat{x}_1$  to downstream agents. Agent 2 consumes up to  $\hat{x}_2$  so the water flow he controls is  $e'_2 = \min\{\hat{x}_2, e_1 - \hat{x}_1\}$ . If  $e'_2 = \hat{x}_2$ , then  $e'_1 - \hat{x}_1 - \hat{x}_2$  is always passed to agent 3 and so on until the whole common pool  $e_1$  is exhausted. Hence, this “lake” sharing problem is equivalent to the river sharing problem where the first  $i$  upstream agents have tributaries  $e'_i = \hat{x}_i$  up to some agent  $h$  whose tributary is  $e'_h = e_1 - \sum_{i \in P^0_h} \hat{x}_i < \hat{x}_h$ . The agents  $j > h$  are endowed with  $e'_j = 0$ .

The two extremes of behavior of the agents outside of  $S$  are the following: either they do not cooperate at all or they all cooperate.

**Non-Cooperative Core Lower Bounds:** For all coalitions  $S$ , let  $\underline{v}(S) = v(S, \{S\} \cup \{\{i\} | i \in N \setminus S\})$ .

**Cooperative Core Lower Bounds:** For all coalitions  $S$ , let  $\bar{v}(S) = v(S, \{S, N \setminus S\})$ .

We say that *cooperation exerts no externality on a coalition  $S$*  if for any partition  $\mathcal{P}$  of  $N$  such that  $S \in \mathcal{P}$ ,

$$\underline{v}(S) = v(S, \mathcal{P}).$$

Then the value of a coalition is independent of the interactions of the other agents. We say that *cooperation exerts a positive externality on a coalition  $S$*  if for any partition  $\mathcal{P}$  of  $N$  such that  $S \in \mathcal{P}$ ,

$$\underline{v}(S) \leq v(S, \mathcal{P}).$$

Then cooperation does not decrease the value of a coalition compared to the value under non-cooperative behavior.

The following proposition contains some basic relations among the core lower bounds of a coalition for different behaviors of its complement. First, cooperation exerts a positive externality on a coalition. Therefore, the non-cooperative core lower bound of a coalition is the smallest core lower bound of all possible behaviors of its complement and non-cooperative behavior of the other agents is the pessimistic expectation for a coalition. This also implies that the non-cooperative core lower bounds are below the cooperative core lower bounds and the cooperative core lower bounds are more demanding than the non-cooperative lower bounds. Finally, the following super-additivity property is true: for any partition of  $N$ , if two coalitions belonging to the partition merge, then their joint payoff does not fall compared to the payoff when they are separate.

**Proposition 1** *Let  $\mathcal{P}$  be partition of  $N$  and  $S \in \mathcal{P}$ .*

(i)  $\underline{v}(S) \leq v(S, \mathcal{P})$ .

(ii)  $\underline{v}(S) \leq \bar{v}(S)$ .

(iii) *For any two disjoint coalitions  $S, T \in \mathcal{P}$ ,  $v(S, \mathcal{P}) + v(T, \mathcal{P}) \leq v(S \cup T, \mathcal{P}')$  where  $\mathcal{P}' = (\mathcal{P} \setminus \{S, T\}) \cup \{S \cup T\}$ .*

**Proof.** Note that (ii) follows from (i). We show (i): let  $\underline{\mathcal{P}} = \{S\} \cup \{\{i\} | i \in N \setminus S\}$  and  $\mathcal{C}(S) = \{S_1, \dots, S_k\}$  where  $S_1 < \dots < S_k$ . Because  $e_i \leq \hat{x}_i$  for all  $i \in N$  and under  $\underline{\mathcal{P}}$  behavior is non-cooperative, we have  $E_{\min S_1}(x^{\underline{\mathcal{P}}}(N)) = 0$  and for all  $l \in \{1, \dots, k-1\}$ ,  $E_{\max S_{l+1}}(x^{\underline{\mathcal{P}}}(N)) \geq E_{\min S_{l+1}}(x^{\underline{\mathcal{P}}}(N))$ , i.e. the agents between any two connected components  $S_l$  and  $S_{l+1}$  consume their peak or the amount of water which is entering the river at their location. Consider a subgame starting at  $S_l$  ( $l \in \{1, \dots, k-1\}$ ) and let  $S_l$  pass the amount  $E' > 0$  of unused water to  $\max S_{l+1}$ . If this is profitable for  $S$  in the subgame starting at  $S_l$  under  $\underline{\mathcal{P}}$ , then all agents between  $S_l$  and  $S_{l+1}$  consume their peak. Under  $\mathcal{P}$ , in the subgame starting at  $S_l$ , each agent between  $S_l$  and  $S_{l+1}$  either consumes his peak or *less*. Therefore, for any

$E' > 0$  which is profitable for  $S$  under  $\underline{\mathcal{P}}$ , under  $\mathcal{P}$  at least the same amount of unused water is passed to  $\min S_{l+1}$  as under  $\underline{\mathcal{P}}$ . Hence, given such an  $E'$ , the set of possible water consumptions which  $S_{l+1}$  can choose under  $\mathcal{P}$  is a superset set of the possible water consumptions which  $S_{l+1}$  can choose under  $\underline{\mathcal{P}}$ . Since the this argument holds for  $S_1$  and  $x^{\mathcal{P}}(N)$  is a Nash equilibrium, we must have  $v(S, \mathcal{P}) \geq v(S, \underline{\mathcal{P}})$ .

We show (iii):  $x^{\mathcal{P}'}(N)$  is the outcome of the backwards induction algorithm under  $\mathcal{P}'$  and therefore,  $x^{\mathcal{P}'}(N)$  is a Nash equilibrium of the dynamic game under perfect information given by the river. If coalition  $S \cup T$  plays alternatively  $x_{S \cup T}^{\mathcal{P}}(N)$ , then the outcome of the backwards induction algorithm is  $x^{\mathcal{P}}(N)$ . Therefore,

$$v(S \cup T, \mathcal{P}') = \sum_{i \in S \cup T} b_i(x_i^{\mathcal{P}'}(N)) \geq \sum_{i \in S \cup T} b_i(x_i^{\mathcal{P}}(N)) = v(S, \mathcal{P}) + v(T, \mathcal{P}),$$

the desired conclusion. □

It is obvious from our definition that the value of a coalition consisting of an agent and his predecessors is independent of how the other agents behave, i.e. for all  $i \in N$  and all  $Pi \in \mathcal{P}$  we have

$$\underline{v}(Pi) = v(Pi, \mathcal{P}) = \bar{v}(Pi).$$

Thus, cooperation exerts no externality on the coalition  $Pi$ . Even though the value of a coalition may depend on how the other agents behave, the structure of the river induces a unique natural incremental distribution, namely the downstream incremental distribution  $z^*$ : for all  $i \in N$ , let

$$z_i^* = \underline{v}(Pi) - \underline{v}(P^0i).$$

## 4 Non-Cooperative Core Lower Bounds and Aspiration Upper Bounds

The aspiration upper bounds are implied by the UTI doctrine. Contrary to the core lower bounds, these upper bounds do not depend on how the agents outside of a

coalition behave. The aspiration welfare of a coalition  $S$  is the highest welfare it could achieve in the absence of  $N \setminus S$ . It is obtained by choosing a consumption plan  $y(S) \in \mathbb{R}_+^S$  maximizing  $\sum_{i \in S} b_i(y_i(S))$  subject to the constraints

$$\sum_{i \in P_j \cap S} y_i(S) \leq \sum_{i \in P_j} e_i \text{ for all } j \in S.$$

Since all benefit functions are strictly concave, the maximization problem has a unique solution, which we denote by  $y^*(S)$ . Then the aspiration welfare of  $S$  is

$$w(S) = \sum_{i \in S} b_i(y_i^*(S)).$$

A distribution  $z$  satisfies the aspiration upper bounds if  $\sum_{i \in S} z_i \leq w(S)$  for all coalitions  $S$ . In the Lockean tradition, coalition  $S$  has a legitimate right to the welfare level  $w(S)$  but not to more. Unfortunately the aspirations of two complementary coalitions  $S$  and  $N \setminus S$  are incompatible:  $w(S) + w(N \setminus S) > \underline{v}(N)$ . It is even the case that for any partition  $\mathcal{P}$  of  $N$  such that  $S \in \mathcal{P}$  we have  $\sum_{T \in \mathcal{P}} w(T) > \underline{v}(N)$ , i.e. the aspiration of  $S$  is never compatible with the aspiration(s) of  $N \setminus S$  independently of how  $N \setminus S$  cooperates. Therefore, if  $\sum_{i \in S} z_i > w(S)$ , then  $\sum_{i \in N \setminus S} z_i < \sum_{T \in \mathcal{P}: T \neq S} w(T)$ . This means that  $S$  benefits from the existence of  $N \setminus S$  while  $N \setminus S$  suffers from the existence of  $S$ . If none of the agents bears any responsibility for the existence of the others, no coalition is ought to enjoy more than its aspiration upper bound.

**Remark 4** Both the ATS and the UTI doctrines are also inspired by Moulin's (1990) group externalities depending on how we define property rights over water. In the absence of the other agents and the water entering the river at their locations, any agent  $i$  enjoys  $\underline{v}(i)$ . Since  $\sum_{i \in N} \underline{v}(i) \leq \underline{v}(N)$ , then our problem has positive group externalities and any agent  $i$  should receive at least  $\underline{v}(i)$ . This inspires the ATS doctrine for individuals and groups. In the absence of the other agents and the presence of the water entering the river at their locations, any agent enjoys  $i$  enjoys  $w(i)$ . Since  $\sum_{i \in N} w(i) \geq \underline{v}(N)$ , then our problem has negative group externalities and any agent  $i$  should receive at most  $w(i)$ . This inspires the UTI doctrine for individuals and groups.

**Remark 5** There is an obvious relation between the non-cooperative core lower bounds and the aspiration upper bounds:  $\underline{v}(Pi) = w(Pi)$  for all  $i \in N$ . Now the following is easy to show: if a distribution satisfies the non-cooperative lower bounds and the aspiration upper bounds, then it must be the downstream incremental distribution.<sup>9</sup>

The main challenge of our paper is to find distributions which satisfy core lower bounds. These bounds depend on the behavior of the agents outside of a coalition. In the case of non-cooperative behavior, there are distributions satisfying the core lower bounds (or belonging to the  $\gamma$ -core). It turns out that in the presence of optimal water consumptions and non-cooperative behavior, the downstream incremental distribution is the only compromise between the ATS and the UTI doctrines.

**Theorem 1** *The downstream incremental distribution is the unique distribution satisfying the non-cooperative core lower bounds and the aspiration upper bounds.*

**Proof.** By Remark 5, if a distribution  $z$  satisfies the non-cooperative core lower bounds and the aspiration upper bounds, then we must have  $z = z^*$ .

Next we show that  $z^*$  satisfies the non-cooperative lower bounds. Let  $S$  be connected and  $\mathcal{P} = \{S\} \cup \{\{i\} | i \in N \setminus S\}$ . Because behavior is non-cooperative, we have for all  $i \in P^0 \min S$ ,  $x_i^{\mathcal{P}}(N) = e_i$ . Thus,  $E_{\min S}(x^{\mathcal{P}}(N)) = 0$ . Since  $S$  is connected,  $\{S, P^0 \min S\}$  is a partition of  $P \max S$ . Hence, by  $E_{\min S}(x^{\mathcal{P}}(N)) = 0$ ,

$$\underline{v}(P \max S) \geq \underline{v}(S) + \underline{v}(P^0 \min S).$$

Thus, for any connected  $S$ ,

$$\sum_{i \in S} z_i^* = \underline{v}(P \max S) - \underline{v}(P^0 \min S) \geq \underline{v}(S). \quad (5)$$

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<sup>9</sup>The proof is identical to Ambec and Sprumont (2002): Let  $z$  be a distribution satisfying the non-cooperative lower bounds and the aspiration upper bounds. Since  $\underline{v}(1) = w(1)$ , we have  $z_1 = \underline{v}(1) = z_1^*$ . Let  $z_i = z_i^*$  for all  $i < j \leq n$ . Since  $\underline{v}(Pj) = w(Pj)$ , we have  $\sum_{i \in Pj} z_i = \underline{v}(Pj)$ . Thus, by  $\sum_{i \in P^0 j} z_i = \sum_{i \in P^0 j} z_i^* = \underline{v}(P^0 j)$ , we obtain  $z_j = \underline{v}(Pj) - \sum_{i \in P^0 j} z_i = \underline{v}(Pj) - \underline{v}(P^0 j) = z_j^*$ , the desired conclusion.

Before we proceed, we note the following: for all  $i \in N$  we have  $\underline{v}(P^0i) + b_i(\hat{x}_i) \geq \underline{v}(Pi)$ . Thus, for all  $i \in N$ ,

$$b_i(\hat{x}_i) \geq \underline{v}(Pi) - \underline{v}(P^0i) = z_i^*. \quad (6)$$

Let  $S$  be an arbitrary coalition and let  $\mathcal{P} = \{S\} \cup \{\{i\} | i \in N \setminus S\}$ . Since  $e_i \leq \hat{x}_i$  for all  $i \in N$ , we have  $\sum_{i \in P^0 \min S} (x_i^{\mathcal{P}}(N) - e_i) = 0$ . Hence,  $E_{\min S}(x^{\mathcal{P}}(N)) = 0$ . Let  $\mathcal{C}(S) = \{S_1, \dots, S_L\}$  where  $S_1 < S_2 < \dots < S_L$ . Choose the minimal  $l \in \{1, \dots, L\}$  such that  $E_{\max S_{l+1}}(x^{\mathcal{P}}(N)) = 0$  and set  $T_1 = \cup_{t=1}^l S_t$ . Then, by  $e_i \leq \hat{x}_i$  for all  $i \in N$ , again we have  $E_{\min S_{l+1}}(x^{\mathcal{P}}(N)) = 0$ . Now choose the  $l' > l$  minimal such that  $E_{\max S_{l'+1}}(x^{\mathcal{P}}(N)) = 0$  and set  $T_2 = \cup_{t=l+1}^{l'} S_t$ . Continuing this way we find a partition  $\mathcal{T} = \{T_1, T_2, \dots, T_M\}$  of  $S$ . By construction,  $T_1 < T_2 < \dots < T_M$  and

$$\underline{v}(S) = \sum_{T \in \mathcal{T}} \underline{v}(T). \quad (7)$$

For each  $T \in \mathcal{T}$ , let  $\bar{T} = P \max T \setminus P^0 \min T$ . Then by definition of  $T$ , we have for all  $i \in \bar{T} \setminus T$ ,  $E_i(x^{\mathcal{P}}(N)) > 0$  and therefore,  $x_i^{\mathcal{P}}(N) = \hat{x}_i$  for all  $i \in \bar{T} \setminus T$ . Now we have

$$\begin{aligned} \sum_{T \in \mathcal{T}} \sum_{i \in \bar{T}} z_i^* &\geq \sum_{T \in \mathcal{T}} \underline{v}(\bar{T}) \\ &\geq \sum_{T \in \mathcal{T}} (\underline{v}(T) + \sum_{i \in \bar{T} \setminus T} b_i(\hat{x}_i)) \\ &= \underline{v}(S) + \sum_{T \in \mathcal{T}} \sum_{i \in \bar{T} \setminus T} b_i(\hat{x}_i), \end{aligned}$$

where the first equality follows from (5) and the fact that each  $\bar{T}$  is connected, the second inequality follows from the fact that  $x_i^{\mathcal{P}}(N) = \hat{x}_i$  for all  $i \in \bar{T} \setminus T$  and the consumption plan  $(x_i^{\mathcal{P}}(N))_{i \in \bar{T}}$  is feasible for  $\bar{T}$ , and the equality follows from (7). Therefore, we have

$$\sum_{i \in S} z_i^* = \sum_{T \in \mathcal{T}} \sum_{i \in T} z_i^* \geq \underline{v}(S) + \sum_{T \in \mathcal{T}} \sum_{i \in \bar{T} \setminus T} (b_i(\hat{x}_i) - z_i^*).$$

From (6) we know that  $b_i(\hat{x}_i) - z_i^* \geq 0$  for all  $i \in N$ . Hence,  $\sum_{i \in S} z_i^* \geq \underline{v}(S)$  and  $z^*$  satisfies the non-cooperative core lower bounds.

The proof that  $z^*$  satisfies the aspiration upper bounds uses the following lemma which we prove in the Appendix.

*Lemma 1* If  $S \subseteq T \subseteq N$  and  $T < i$ , then  $w(S \cup i) - w(S) \geq w(T \cup i) - w(T)$ .

Then for any coalition  $S$  we obtain

$$\sum_{i \in S} z_i^* = \sum_{i \in S} (w(Pi) - w(P^0i)) \leq \sum_{i \in S} (w(Pi \cap S) - w(P^0i \cap S)) = w(S),$$

where the inequality follows from Lemma 1 and the last equality follows from the fact that all terms cancel out except  $w(P \max S \cap S) = w(S)$  and  $-w(P^0 \min S \cap S) = w(\emptyset) = 0$ .  $\square$

**Remark 6** It can be easily checked that Theorem 1 and its proof remain true if agents are allowed to have benefit functions which either have a satiation point or are strictly increasing (as in Ambec and Sprumont (2002)). Therefore, Theorem 1 generalizes the theorem of Ambec and Sprumont (2002). In the presence of satiation points the main difference and (non-trivial) difficulty is to show that the downstream incremental distribution satisfies the non-cooperative core lower bounds. In Ambec and Sprumont (2002) this was straightforward because with strictly increasing benefit functions it is never optimal for a coalition to pass water from one component to another and cooperation exerts no externality on any coalition. Therefore, their game is consecutive (Greenberg and Weber, 1986) meaning that the value of a coalition equals the sum of the values of its connected components. Then for showing that a distribution satisfies the core lower bounds, it is sufficient to show that the distribution satisfies the core lower bounds for connected coalitions.

**Remark 7** The grand coalition  $N$  needs not necessarily to form in order to implement the efficient allocation and downstream incremental distribution. Instead of having a global agreement, local agreements among coalitions belonging to the “efficient” partition  $\{N_k\}_{k=1, \dots, K}$  (defined by conditions (1) to (4)) equivalently implement both the optimal consumption plan  $x^*(N)$  and the downstream incremental distribution  $z^*$ . It is straightforward that the consumption plan  $(x_i^*(N))_{i \in N_k}$  of each subset

$N_k$  coincides with the optimal consumption plan for the subriver sharing problem  $(N_k, b_{N_k}, e_{N_k})$ , i.e. the portion  $N_k$  of the river  $N$ . Moreover, for any  $N_k$  it is easy to show from (1) to (4) that  $(z_i^*)_{i \in N_k}$  is the downstream incremental distribution of the subriver sharing problem  $(N_k, b_{N_k}, e_{N_k})$ .<sup>10</sup>

## 5 (Cooperative) Core Lower Bounds

The downstream incremental distribution satisfies the non-cooperative core lower bounds. We investigate whether there exist distributions satisfying the core lower bounds when agents cooperate, i.e. once a coalition  $S$  forms the complement  $N \setminus S$  can also form coalitions. First, we focus on the other extreme of non-cooperative behavior, namely on cooperative behavior.

Contrary to non-cooperative behavior, the downstream incremental distribution may violate the cooperative core lower bounds when there are at least three agents. Note that for two agents we have  $\underline{v} = \bar{v}$  and the non-cooperative core lower bounds and the cooperative core lower bounds are identical.

**Example 1** (*the downstream incremental distribution may violate the cooperative core lower bounds*) Let  $N = \{1, 2, 3\}$ ,  $e_1 = e_2 = \frac{5}{6}$ ,  $e_3 = 0$ , and  $b_1 = b_2$  and  $b_3 = 100b_1$  be such that  $\hat{x}_1 = \hat{x}_2 = \hat{x}_3 = 1$ . Then  $x_1^{\{\{1,2\},\{3\}\}}(N) = x_2^{\{\{1,2\},\{3\}\}}(N) = \frac{5}{6}$  and  $z_2^* = \underline{v}(\{1, 2\}) - \underline{v}(1) = b_2(\frac{5}{6})$ . Consider  $\{2\}$  and  $\{\{2\}, \{1, 3\}\}$ . Since  $b_3 = 100b_1$ , it is obvious that  $b_1(\frac{5}{6}) + b_3(0) < b_1(0) + b_3(\frac{4}{6})$ . Therefore, coalition  $\{1, 3\}$  chooses to pass water from 1 to 3 and we have both  $E_2(x^{\{\{2\},\{1,3\}\}}(N)) > 0$  and  $\frac{5}{6} < x_2^{\{\{2\},\{1,3\}\}}(N) \leq 1$ . Hence,  $\bar{v}(2) = b_2(x_2^{\{\{2\},\{1,3\}\}}(N)) > b_2(\frac{5}{6}) = z_2^*$  and the downstream incremental distribution  $z^*$  violates the cooperative core lower bounds.

It is easy to extend Example 1 for  $n > 3$ . Now one may wonder whether other

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<sup>10</sup>It follows from (1) to (4) that  $b'_{\min N_2}(e_{\min N_2}) \leq \beta_2$  and (where  $P^0 \min N_2 = N_1$ )  $\underline{v}(P \min N_2) = \underline{v}(N_1) + \underline{v}(\min N_2)$ . Therefore,  $z_{\min N_2}^* = \underline{v}(\min N_2)$  which is identical with agent  $\min N_2$ 's value in the subriver sharing problem  $(N_2, b_{N_2}, e_{N_2})$ . The same argument applies to any  $N_k$ .

distributions satisfy the cooperative core lower bounds. For three agents, the answer is positive and the cooperative core is non-empty.

**Proposition 2** *For  $N = \{1, 2, 3\}$  there exists always a distribution satisfying the cooperative core lower bounds.*

**Proof.** Let  $n = 3$ . If  $\bar{v}(2) \leq \underline{v}(1, 2) - \underline{v}(2)$  then, since  $\underline{v}(Pi) = \bar{v}(Pi)$  and  $\underline{v}(Fi) = \bar{v}(Fi)$  for every  $i \in N$  and  $\underline{v}(1, 3) = \bar{v}(1, 3)$ , then the downstream incremental distribution  $z^*$  satisfies the cooperative core lower bounds.

Suppose now that  $\bar{v}(2) \geq \underline{v}(1, 2) - \underline{v}(1)$ . We show that the distribution  $z_1 = \bar{v}(1)$ ,  $z_2 = \bar{v}(2)$ ,  $z_3 = \bar{v}(1, 2, 3) - \bar{v}(1) - \bar{v}(2)$  satisfies the cooperative core lower bounds. First, note that  $z_3 = \bar{v}(3) + \bar{v}(1, 2, 3) - \bar{v}(1) - \bar{v}(2) - \bar{v}(3)$  and  $\bar{v}(1) + \bar{v}(3) = \underline{v}(1) + \underline{v}(3) \leq \underline{v}(1, 3) = \bar{v}(1, 3)$ . These two relationships imply  $z_3 \geq \bar{v}(3) + \bar{v}(1, 2, 3) - \bar{v}(1, 3) - \bar{v}(2)$  which leads to  $z_3 \geq \bar{v}(3)$  because  $\bar{v}(1, 2, 3) \geq \bar{v}(1, 3) + \bar{v}(2)$ . Second, substituting our initial assumption  $\bar{v}(2) \geq \underline{v}(1, 2) - \underline{v}(1)$  into  $z_1 + z_2 = \bar{v}(1) + \bar{v}(2)$  shows that  $z_1 + z_2 \geq \underline{v}(1, 2) = \bar{v}(1, 2)$ . Third,  $z_1 + z_3 = \bar{v}(1, 2, 3) - \bar{v}(2) \geq \bar{v}(1, 3)$  because  $\bar{v}(1, 2, 3) \geq \bar{v}(1, 3) + \bar{v}(2)$ . Similarly  $z_2 + z_3 = \bar{v}(1, 2, 3) - \bar{v}(1) \geq \bar{v}(2, 3)$  because  $\bar{v}(1, 2, 3) \geq \bar{v}(2, 3) + \bar{v}(1)$ . The other cooperative core lower bounds are obviously satisfied.  $\square$

Unfortunately Proposition 2 is true only for three agents. When the downstream incremental distribution violates the cooperative core lower bounds, for more than three agents there may not exist another (alternative) distribution satisfying the cooperative core lower bounds. Therefore, similarly as in Carraro and Siniscalco (1993) the cooperative or  $\delta$ -core may be empty for more than three agents.

**Theorem 2** *When there are more than three agents all distributions may violate the cooperative core lower bounds.*

The following example establishes Theorem 2.

**Example 2** (*The cooperative core may be empty*) Let  $N = \{1, 2, 3, 4\}$  and the benefit functions  $b$  be such that  $b_1(x) = 50x - \frac{x^2}{2}$  for all  $x \in [20, 100]$ ,  $b_2(x) = b_3(x) = 100x - 10x^2$  for all  $x \in [3, 10]$  and  $b_4(x) = 2b_1(x)$ . The river inflows are  $e_1 = 33$ ,  $e_2 = e_3 = 4$ ,  $e_4 = 37$ .

We show that  $\bar{v}(2) = b_2(\hat{x}_2)$ ,  $\bar{v}(3) = b_3(\hat{x}_3)$ , and  $\bar{v}(1, 2, 3, 4) < \underline{v}(1) + \bar{v}(2) + \bar{v}(3) + \underline{v}(4)$ . The last condition implies that no distribution satisfies each agent  $i$ 's cooperative core lower bound  $\bar{v}(i)$  (note that  $\underline{v}(1) = \bar{v}(1)$  and  $\underline{v}(4) = \bar{v}(4)$ ).

First, the optimal consumption plan  $x^*(N)$  solves the maximization program defined by  $\bar{v}(1, 2, 3, 4)$ . Because  $b_4 = 2b_1$ ,  $x^*(N)$  equalizes agents' marginal benefits, i.e.,  $50 - x_1^*(N) = 100 - 20x_2^*(N) = 100 - 20x_3^*(N) = 100 - 2x_4^*(N)$ , and satisfies the global resource constraint  $x_1^*(N) + x_2^*(N) + x_3^*(N) + x_4^*(N) = e_1 + e_2 + e_3 + e_4 = 78$ . The solution is  $(30, 4, 4, 40)$ . Therefore  $\bar{v}(1, 2, 3, 4) = b_1(30) + b_2(4) + b_3(4) + b_4(40) = 3930$ .

Second, we show that coalition  $\{1, 3, 4\}$  chooses to pass three units of water from 1 to  $\{3, 4\}$ . Therefore, 2 consumes  $\hat{x}_2 = 5$  units of water and  $\bar{v}(2) = b_2(\hat{x}_2)$ . Doing so, coalition  $\{1, 3, 4\}$  loses  $\hat{x}_2 - e_2 = 1$  unit of water (which is consumed by 2) and 1, 3, and 4, respectively, can consume 30, 4, and 39 units of water. The welfare achieved is  $b_1(30) + b_3(4) + b_4(39) = 3690$ . If no water is passed from 1 to  $\{3, 4\}$ , then 1 consumes  $e_1 = 33$  and 3 and 4 share optimally 41 units of water by consuming respectively  $\frac{41}{11}$  and  $\frac{410}{11}$ . The welfare is then  $b_1(33) + b_3(\frac{41}{11}) + b_4(\frac{410}{11}) \approx 3677.32 < 3690$ .

Third, we show that the coalition  $\{1, 2, 4\}$  chooses to pass three units of water from  $\{1, 2\}$  to 4. Therefore, 3 consumes  $\hat{x}_3 = 5$  units of water and  $\bar{v}(3) = b_3(\hat{x}_3)$ . Doing so, coalition  $\{1, 2, 4\}$  loses  $\hat{x}_3 - e_3 = 1$  unit of water (which is consumed by 3) and 1 and 2, respectively, can consume 30 and 4 units whereas 4 consumes  $e_4 + 2 = 39$  units. The welfare achieved is  $b_1(30) + b_2(4) + b_4(39) = 3690$ . If no water is passed from  $\{1, 2\}$  to 4, then 1 and 2 share optimally  $e_1 + e_2 = 37$  units of water by consuming respectively (approximatively) 32.9 and 4.1 and 4 consumes  $e_4 = 37$ . The welfare is then  $b_1(32.9) + b_3(4.1) + b_4(37) \approx 3676.7 < 3690$ .

Finally,  $\underline{v}(1) + \bar{v}(2) + \bar{v}(3) + \underline{v}(4) = b_1(e_1) + b_2(\hat{x}_2) + b_3(\hat{x}_3) + b_4(e_4) = b_1(33) +$

$b_2(5) + b_3(5) + b_4(37) = 3936.5 > 3930 = \bar{v}(1, 2, 3, 4)$ . Hence, all distributions violate the cooperative core lower bounds.

Given Theorem 2, next we investigate when there exist distributions satisfying the core lower bounds. When the expectations of any coalition are pessimistic, the downstream incremental distribution satisfies the core lower bounds. When the expectations of any coalition are optimistic, there may not be any distribution satisfying the core lower bounds. This is even true if blocking is restricted to connected coalitions (see Example 2). Similarly to Demange (2004) allowing blocking only for connected coalitions is natural for a river because cooperation is easier for connected coalitions. The following result gives a simple necessary and sufficient condition for the existence of a compromise between the UTI doctrine and the ATS doctrine for all connected coalitions under optimistic expectations.

We will say that a distribution  $z$  satisfies for any connected coalition  $S$  all core lower bounds if  $\sum_{i \in S} z_i \geq v(S, \mathcal{P})$  for all partitions  $\mathcal{P}$  of  $N$  such that  $S \in \mathcal{P}$ . Then the core lower bound of  $S$  is always satisfied independently of how the agents outside of  $S$  are organized.

**Theorem 3** *The following are equivalent:*

- (i) *There exists a distribution satisfying the aspiration upper bounds and for any connected coalition all core lower bounds.*
- (ii) *The downstream incremental distribution satisfies for any connected coalition all core lower bounds.*
- (iii) *Cooperation exerts no externality on the value of any agent, i.e.  $\underline{v}(i) = v(i, \mathcal{P})$  for all  $i \in N$  and all partitions  $\mathcal{P}$  of  $N$  such that  $\{i\} \in \mathcal{P}$ .*

**Proof.** (i)  $\Rightarrow$  (ii): Let  $z$  be a distribution satisfying the aspiration upper bounds and for any connected coalition all core lower bounds. By Proposition 1, all core lower bounds are greater than or equal to the non-cooperative core lower bounds. Hence,

$z$  is a distribution satisfying the aspiration upper bounds and for any connected coalition the non-cooperative core lower bounds. Then, by Remark 5, we have  $z = z^*$  and the downstream incremental distribution satisfies for any connected coalition all core lower bounds.

(ii)⇒(i): By Theorem 1,  $z^*$  satisfies the aspiration upper bounds. Hence,  $z^*$  is a distribution satisfying the aspiration upper bounds and for any connected coalition all core lower bounds.

(ii)⇒(iii): Let  $z^*$  satisfy all core lower bounds for all connected coalitions. Let  $i \in N$  and  $\mathcal{P}$  be a partition of  $N$  such that  $\{i\} \in \mathcal{P}$ . Since  $\{i\}$  is connected, we have  $\underline{v}(Pi) - \underline{v}(P^0i) = z_i^* \geq v(i, \mathcal{P})$ . Hence, by  $v(i, \mathcal{P}) = b_i(x_i^{\mathcal{P}}(N))$ ,

$$\underline{v}(Pi) \geq \underline{v}(P^0i) + b_i(x_i^{\mathcal{P}}(N)). \quad (8)$$

On the other hand, by  $\sum_{j \in P^0i} x_j^{\{P^0i, N \setminus P^0i\}}(N) = \sum_{j \in P^0i} e_j \geq \sum_{j \in P^0i} x_j^{\{Pi, N \setminus Pi\}}(N)$ ,  $x_{P^0i}^{\{Pi, N \setminus Pi\}}(N)$  is a consumption plan for  $P^0i$ . Therefore,  $\underline{v}(P^0i) \geq \sum_{j \in P^0i} b_j(x_j^{\{Pi, N \setminus Pi\}}(N))$  and

$$\underline{v}(P^0i) + b_i(x_i^{\{Pi, N \setminus Pi\}}(N)) \geq \underline{v}(Pi). \quad (9)$$

Hence, from (8) and (9) we obtain  $b_i(x_i^{\{Pi, N \setminus Pi\}}(N)) \geq b_i(x_i^{\mathcal{P}}(N))$ . Since agent  $i$ 's consumption is always smaller than or equal to  $\hat{x}_i$  and  $b_i$  is strictly increasing between 0 and  $\hat{x}_i$ , the previous inequality is equivalent to

$$x_i^{\{Pi, N \setminus Pi\}}(N) \geq x_i^{\mathcal{P}}(N) \quad (10)$$

By  $\{i\} \in \mathcal{P}$ , we have  $x_i^{\mathcal{P}}(N) \in \{e_i, \hat{x}_i\}$ . If  $x_i^{\mathcal{P}}(N) = e_i$ , then  $v(i, \mathcal{P}) = b_i(e_i) = \underline{v}(i)$ , the desired conclusion. If  $x_i^{\mathcal{P}}(N) \neq e_i$ , then  $x_i^{\mathcal{P}}(N) = \hat{x}_i$ . Hence, by (10),  $x_i^{\{Pi, N \setminus Pi\}}(N) = \hat{x}_i$ , and by  $\hat{x}_i \geq e_i$ ,  $\hat{x}_i > e_i$ . But then, by  $x_i^{\{Pi, N \setminus Pi\}}(N) = \hat{x}_i > e_i$ , we have

$$\sum_{j \in P^0i} x_j^{\{Pi, N \setminus Pi\}}(N) < \sum_{j \in P^0i} e_j.$$

Therefore,

$$\sum_{j \in P^0i} b_j(x_j^{\{Pi, N \setminus Pi\}}(N)) < \underline{v}(P^0i). \quad (11)$$

Hence,

$$\underline{v}(Pi) = \sum_{j \in P^0i} b_j(x_j^{\{Pi, N \setminus Pi\}}(N)) + b_i(x_i^{\{Pi, N \setminus Pi\}}(N)) < \underline{v}(P^0i) + b_i(\hat{x}_i),$$

where the inequality follows from (11) and  $x_i^{\{Pi, N \setminus Pi\}}(N) = \hat{x}_i$ . Now, by  $x_i^{\mathcal{P}}(N) = \hat{x}_i$ , this inequality contradicts (8). Thus, we have to have  $x_i^{\mathcal{P}}(N) = e_i$  and  $v(i, \mathcal{P}) = \underline{v}(i)$  for all  $i \in N$  and all  $\mathcal{P}$  such that  $\{i\} \in \mathcal{P}$ .

(iii)  $\Rightarrow$  (ii): Let  $S$  be a connected coalition and  $\mathcal{P}$  be a partition such that  $S \in \mathcal{P}$ . We show  $\underline{v}(S) = v(S, \mathcal{P})$ . Since  $S$  is connected, we have either  $\underline{v}(S) = v(S, \mathcal{P})$  or  $\underline{v}(S) < v(S, \mathcal{P}) = \sum_{i \in S} b_i(\hat{x}_i)$ . Suppose that

$$\underline{v}(S) < v(S, \mathcal{P}) = \sum_{i \in S} b_i(\hat{x}_i). \quad (12)$$

Then there exists  $i \in S$  such that  $e_i < \hat{x}_i$ . Let  $\mathcal{P}' = (\mathcal{P} \setminus S) \cup \{\{j\} | j \in S\}$ . By (12),  $x_j^{\mathcal{P}}(N) = \hat{x}_j$  for all  $j \in S$ . Then  $x^{\mathcal{P}}(N)$  is also the outcome of the backwards induction algorithm when agents cooperate according to  $\mathcal{P}'$ . Hence,  $x_j^{\mathcal{P}'}(N) = \hat{x}_j$  for all  $j \in S$  and  $v(i, \mathcal{P}') = b_i(\hat{x}_i)$ . Since  $e_i < \hat{x}_i$  and  $\underline{v}(i) = b_i(e_i)$ , we obtain  $\underline{v}(i) < v(i, \mathcal{P}')$ , which contradicts (ii). Hence, (12) was wrong and we have  $\underline{v}(S) = v(S, \mathcal{P})$ .

By Theorem 1,  $z^*$  satisfies the non-cooperative core lower bounds. Hence, by  $\underline{v}(S) = v(S, \mathcal{P})$  for all connected coalitions  $S$  and all partitions  $\mathcal{P}$  such that  $S \in \mathcal{P}$ ,  $z^*$  satisfies for any connected coalition all core lower bounds, the desired conclusion.  $\square$

By Theorem 3, the downstream incremental distribution satisfies all core lower bounds for all connected coalitions if and only if the the individual rationality constraints are identical under all behaviors of the other agents. Condition (iii) of Theorem 3 is trivially satisfied in Ambec and Sprumont (2002) because in their paper no coalition is passing water from one of its connected components to another one independently of the behavior of the other agents. Basically, it requires that no agent can “free ride” on the other agents’ cooperative behavior. Such a condition can be estimated in real-world river sharing problems.

**Remark 8** The equivalence in Theorem 3 does not remain true under cooperative behavior, i.e. if we require that the cooperative core lower bounds are satisfied for any connected coalition. Below we provide an example showing that  $\underline{v}(i) = \bar{v}(i)$  for all  $i \in N$  but the downstream incremental distribution violates a cooperative core lower bound for a connected coalition.

By Proposition 1, cooperation exerts a positive externality on a coalition (compared to non-cooperative behavior). Then one may wonder whether starting from any partition “more” cooperation of the other agents always exerts a positive externality on the value of a coalition. Here “more” cooperation means that from a partition we obtain a coarser partition by merging some coalitions. If this were true, then the cooperative core lower bound of a coalition is maximal among all core lower bounds for all behaviors of the other agents. The following example shows that merging of some coalitions may exhibit a negative externality on the value of a coalition (compared to the value of the coalition before the merger).<sup>11</sup>

**Example 3** (*For a coalition the cooperative core lower bound may not be maximal among all core lower bounds*) Let  $N = \{1, 2, 3, 4\}$  and the benefit functions  $b$  be such that  $b_1(x) = 50x - \frac{x^2}{2}$  for all  $x \in [20, 100]$ ,  $b_2(x) = b_3(x) = 100x - 10x^2$  for all  $x \in [3, 10]$  and  $b_4(x) = 2b_1(x)$ . The river inflows are  $e_1 = 33$ ,  $e_2 = 4$ ,  $e_3 = \hat{x}_3 = 5$ ,  $e_4 = 35$ .

We show the following:  $\underline{v}(i) = \bar{v}(i)$  for all  $i \in N$  and  $\bar{v}(2) < v(2, \{\{1, 4\}, \{2\}, \{3\}\}) = b_2(\hat{x}_2)$ . Therefore, if coalitions  $\{1, 4\}$  and  $\{3\}$  merge, then  $\{2\}$  obtains strictly less than  $v(2, \{\{1, 4\}, \{2\}, \{3\}\})$ . The welfare achieved by a coalition might decrease with a coarser partition of its complement.

First, we show that  $\{1, 4\}$  passes some water from 1 to 4 and 2 consumes his peak  $\hat{x}_2$  under the partition  $\{\{1, 4\}, \{2\}, \{3\}\}$ , i.e.  $v(2, \{\{1, 4\}, \{2\}, \{3\}\}) = b_2(\hat{x}_2)$ . Without passing water, the welfare achieved by  $\{1, 4\}$  is  $b_1(e_1) + b_4(e_4) = b_1(33) +$

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<sup>11</sup>This is in contrast to industrial environments where collusive agreements or cartels reduce market competition or R&D agreements with spillovers.

$b_4(35) = \frac{6761}{2} = 3380.5$ . By passing some water, the coalition loses 1 unit of water (consumed by 2 because  $\hat{x}_2 - e_2 = 1$ ), and agents 1 and 4 share optimally  $33+35-1=67$ . They equalize marginal benefits  $50 - x_1 = 100 - 2x_4$  and satisfy the resource constraint  $x_1 + x_4 = 67$ . Thus, 1 and 4, respectively, consume 28 and 39. Their welfare is  $b_1(28) + b_4(39) = 3387 > 3380.5$ .

Second, we show that if 3 joins the coalition  $\{1, 4\}$ , then coalition  $\{1, 3, 4\}$  chooses not to pass any water from 1 to  $\{3, 4\}$ . Doing so 3 and 4 share optimally  $e_3 + e_4$  and they consume respectively  $\frac{40}{11}$  and  $\frac{400}{11}$ . Then the welfare achieved by  $\{1, 3, 4\}$  is  $b_1(33) + b_3(\frac{40}{11}) + b_4(\frac{400}{11}) = \frac{80321}{22} > 3650$ . If some water is passed from 1 to  $\{3, 4\}$ , then  $e_1 + e_3 + e_4 - 1 = 72$  units of water are shared optimally between the members of  $\{1, 3, 4\}$ . Agents 1, 3 and 4, respectively, consume  $\frac{890}{31}$ ,  $\frac{122}{31}$  and  $\frac{1220}{31}$ . The welfare is then  $b_1(\frac{890}{31}) + b_3(\frac{122}{31}) + b_4(\frac{1220}{31}) = \frac{113110}{31} < 3649$ . Therefore,  $\{1, 3, 4\}$  chooses not to pass any water from 1 to  $\{3, 4\}$  and  $\bar{v}(2) = b_2(e_2) = \underline{v}(2)$ .

Since  $e_3 = \hat{x}_3$ , we have  $\underline{v}(i) = \bar{v}(i)$  for all  $i \in N$ . Furthermore, by  $e_2 < \hat{x}_2$ ,  $\bar{v}(2) < v(2, \{\{1, 4\}, \{2\}, \{3\}\})$ .

Finally, we show that the downstream incremental distribution violates the cooperative core lower bounds although we have  $\underline{v}(i) = \bar{v}(i)$  for all  $i \in N$ . Since  $e_2 < \hat{x}_2$  and  $e_3 = \hat{x}_3$ , we obtain

$$\bar{v}(2, 3) = v(2, \{\{1, 4\}, \{2\}, \{3\}\}) + v(3, \{\{1, 4\}, \{2\}, \{3\}\}) = b_2(\hat{x}_2) + b_3(\hat{x}_3) > \underline{v}(2, 3).$$

Hence, by Theorem 3,  $z^*$  violates the cooperative core lower bounds.

**Remark 9** The analogue of the downstream incremental distribution is the upstream incremental distribution  $u^*$  defined by  $u_i^* = \underline{v}(Fi) - \underline{v}(F^0i)$  (here again it does not matter how the agents in  $F^0i$  behave). One may wonder why it is the downstream incremental distribution which is the most important distribution for our problems (and why not others, for example  $u^*$  which is the distribution corresponding to the “hierarchical outcome” considered by Demange (2004)). One possible explanation is the characterization (1)-(4) of optimal consumption plans. Here, the marginal

benefits at this plan are weakly decreasing downstream meaning that agents more downstream are closer to their optimal consumption. Of course, the same is true for all coalitions. Therefore, distributions which satisfy core lower bounds must put more “importance” on agents who are more downstream.

## APPENDIX.

**Lemma 1** *If  $S \subseteq T \subseteq N$  and  $T < i$ , then  $w(S \cup i) - w(S) \geq w(T \cup i) - w(T)$ .*

**Proof.** The proof follows Ambec and Sprumont (2002). As a first step in the proof of this lemma, let us show that if  $\emptyset \neq S \subset T \subset N$ , then  $y^*(S) \geq y_S^*(T)$ . Clearly, it suffices to establish that  $y^*(S) \geq y_S^*(S \cup t)$  whenever  $\emptyset \neq S \neq N$  and  $t \in N \setminus S$ . Write  $y^*(S) = x$  and  $y_S^*(S \cup t) = y$ . All agents under consideration in the argument belong to  $S$ . By definition of  $x$  and  $y$ ,  $\sum_{i \in S} (y_i - x_i) \leq 0$ . Let  $i_1 \leq \dots \leq i_L$  be those  $i$  such that  $x_i \neq y_i$  (if none exists, there is nothing to prove). We claim that  $y_{i_L} - x_{i_L} < 0$ . Suppose, by contradiction, that the opposite (necessarily strict) inequality is true. Let  $j$  be the largest predecessor of  $i_L$  such that  $y_j - x_j < 0$  (which necessarily exists). Moreover,  $y_j < \hat{x}_j$  since  $x_j \leq \hat{x}_j$ . Define  $y_{i_L}^\varepsilon = y_{i_L} - \varepsilon$ ,  $y_j^\varepsilon = y_j + \varepsilon$ ,  $y_i^\varepsilon = y_i$  for  $i \neq i_L, j$ . Since  $b'_j(y_j) > b'_j(x_j) \geq b'_{i_L}(x_{i_L}) > b'_{i_L}(y_{i_L})$ , choosing  $\varepsilon > 0$  small enough (in particular such that  $y_j + \varepsilon < \hat{x}_j$ ) ensures that  $\sum_{i \in S} (b_i(y_i^\varepsilon) - b_i(y_i)) > 0$  while  $y^\varepsilon$  meets the same constraints as  $y$ , a contradiction. Because  $y_{i_L} - x_{i_L} < 0$ , it now follows that  $y_{i_l} - x_{i_l} < 0$  successively for  $l = L - 1, \dots, 1$ .

Moving to the second step, let  $S \subset T \subset N$  and  $T < i$ . Define  $x'_i = y_i^*(T \cup i)$  and  $x'_j = y_j^*(T \cup i) + y_j^*(S) - y_j^*(T)$  for  $j \in S$ . By our first step,  $y_j^*(T \cup i) \leq y_j^*(T) \leq y_j^*(S)$  for all  $j \in S$ . Therefore  $0 \leq y_j^*(T \cup i) \leq x'_j \leq y_j^*(S)$  for all  $j \in S$  and the consumption plan  $x'$  for  $S \cup i$  satisfies the same constraints as  $y^*(S \cup i)$ , namely,  $\sum_{k \in P_j \cap (S \cup i)} x'_k \leq \sum_{k \in P_j} e_k$  for all  $j \in S \cup i$ . Hence,  $w(S \cup i) \geq \sum_{j \in S \cup i} b_j(x'_j)$  and

$$w(S \cup i) - w(S) \geq b_i(x'_i) + \sum_{j \in S} [b_j(x'_j) - b_j(y_j^*(S))]. \quad (13)$$

On the other hand, since  $y_j^*(T \cup i) \leq y_j^*(T)$  for all  $j \in T \setminus S$ ,

$$w(T \cup i) - w(T) \leq b_i(x'_i) + \sum_{j \in S} [b_j(y_j^*(T \cup i)) - b_j(y_j^*(T))]. \quad (14)$$

Since  $x'_j - y_j^*(S) = y_j^*(T \cup i) - y_j^*(T)$  and  $y_j^*(T \cup i) \leq x'_j$  for all  $j \in S$ , it follows from (13), (14), and the concavity of the benefit functions on its increasing part that  $w(T \cup i) - w(T) \leq w(S \cup i) - w(S)$ . This completes the proof of the lemma.  $\square$

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