Revisiting Consistency in House Allocation Problems and the Computational Approach to the Axiomatic Method*

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Abstract

How should we allocate a social endowment of objects among a group of agents when monetary compensation is not possible? Typical examples are the assignment of offices among faculty and the assignment of tasks among subalterns. We follow an axiomatic approach and impose two properties on rules: Pareto-efficiency and consistency. Previous research arrived at a "dictatorship" result imposing another property: neutrality (Ergin JME, 2000.) We identify Pareto-efficient and consistent rules that depart from this dictatorship and recover some fairness. Under the mild assumption that there are at least four objects, we characterize such a family of rules using minimal notions of Pareto-efficiency and consistency. We investigate the restrictions imposed by strategy-proofness on this family of rules.

As a technical contribution, we develop a "computational" approach to the axiomatic method, which overcomes the challenges entailed in our main characterization.

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1 Introduction

How should we allocate a social endowment of objects among a group of agents when monetary compensation is not possible? Typical examples are the assignment of offices among faculty and the assignment of tasks among subalterns. We study problems, or "economies", where the number of agents and objects are equal and where each agent must receive one object. These problems are known as "house allocation problems".

We consider a variable population setting; we are interested in making recommendations for all possible economies with agent sets coming from a set of "potential agents", and endowed with sets of objects coming from a set of "potential objects". An allocation rule, hereafter referred to as a rule, is a function that recommends one allocation for each economy.

We follow an axiomatic approach and impose two properties on rules: Pareto-efficiency and consistency. Pareto-efficiency, requires that there should not be a way to make at least one agent better off without making some other agent worse off.

Consistency, requires the recommendation of each economy to be stable under recalculation for subgroups of agents. It is motivated by the following thought experiment. Consider the allocation recommended for some economy, and suppose that some agents receive their allotment and "leave" (they may have arrived first to claim it, or been asked to perform a task that precedes the other in a project.) What is the recommendation made by the rule for the problem of assigning the remaining objects among the agents who have not received their allotment yet? Consistency guarantees that the recommendation for this "reduced" problem is identical to the restriction of the initial recommendation to these agents (see Thomson [13] for a survey on consistency.)

Previous research arrived at a "dictatorship" result by imposing another property: neutrality (Ergin [8].) This property requires rules to treat objects "symmetrically". We investigate the class of Pareto-efficient and consistent rules without imposing any other requirement, in particular neutrality. Two main issues motivate our study.

First, fairness considerations. Let A and B be two agents. A rule gives priority to A over B in one particular economy, if the recommendation for such an economy is such that A prefers her allotment to B's and B prefers A's allotment to hers. If a Pareto-efficient and consistent rule satisfies neutrality, then either there are no economies where the rule gives priority to A over B or there are no economies where it gives priority to B over A. On the other hand, there are Pareto-efficient, consistent, and non-neutral rules that give priority to A over B in as many economies as the ones at which B has priority over A. In this sense, departing from dictatorship allows for the recovery of some form of symmetric treatment of the agents.

Second, there are situations where it is desirable an asymmetric treatment of the objects. If there is an allocation at which each agent receives her preferred object, we recommend it. Otherwise, it becomes necessary to choose which agents receive their preferred object. In such a case we may use the information we have about the different objects in order to decide which agents are favored. For instance, we may favor scholars when assigning tasks that require more "brain", and athletes when assigning tasks that require more "brawn".

We identify Pareto-efficient and consistent rules departing from dictatorship. How large is the class of such rules? This is the matter that this paper will explore. To answer this question, we provide a characterization of Pareto-efficient and consistent rules when there are at least four objects.

Pareto-efficient, consistent, and neutral rules are characterized as serial dictatorship rules (Ergin [8].) Serial dictatorship rules recommend allocations based on a linear order of the set of agents. The serial dictatorship rule associated with such an order is defined through the following algorithm. Consider an economy. The first agent in the order receives her preferred object. The second agent receives her preferred object among the remaining objects, and so on.

There are Pareto-efficient and consistent rules that are not serial dictatorships. Our first example is the family of "mixed dictator-pairwise-exchange" (MDPE) rules (Ehlers [7].) These rules generalize serial dictatorships. They recommend allocations based on a partition of the set of potential agents into groups of, at most, two agents and a linear order of the components of this partition. Similar to a serial dictatorship, each agent receives her preferred object given the allotments assigned to agents who belong to groups ranked higher in the order. If two agents who belong to the same group prefer the same object, a "tie-breaker" determines the agent who receives the object in "dispute." This tie-breaker is a function of the agents in conflict and the object in dispute, and disregards any other data.

MDPE rules could be described using an algorithm based on a set of "exogenous priorities" (Ehlers and Klaus [4]; and Ergin [9].) These priorities emerge naturally in "school admission problems", and have been studied in recent literature (Balinski and Sönmez [1]; Ergin [9]; and Kesten [10].)

How far can we go from MDPE rules without giving up Pareto-efficiency and consistency? If the set of potential agents has cardinality two, every Pareto-efficient rule is consistent (not necessarily an MDPE rule.) Moreover, if the set of potential agents and the set of potential objects have cardinality three, there is a Pareto-efficient and consistent rule such that each agent receives her preferred object in the same number of two-agent economies (this is not possible for an MDPE rule, see Example 8.) But, if there are at least four objects, Pareto-efficiency and consistency bring us back almost to MDPE rules.

Our main theorem is a characterization of the family of Pareto-efficient and consistent rules when there are at least four objects. These rules are a "minimal departure" from MDPE rules; they differ from MDPE rules in that they allow for a different treatment of only one group of agents; this group has, at most, three agents and is ranked last with respect to the order of the components of the set of potential agents' partition (see Section 3 for details.) We call these rules "mixed dictator-pairwise-exchange with tail" (MDPEwt) rules.

A weaker notion of Pareto-efficiency is unanimity. This property requires rules to choose the allocation at which each agent receives her preferred object, if there is such an allocation. If a rule is unanimous and consistent, then it is Pareto-efficient (Takamiya [12].) We further the understanding of the logical relations between Pareto-efficiency, unanimity, and consistency by defining a family of weaker notions of unanimity and consistency indexed by natural numbers: n-unanimity and n-consistency; n-unanimity restricts the requirements of unanimity to economies with n or fewer agents; n-consistency requires the recommendation of each economy to be stable under recalculation for subgroups with n or fewer agents. We present our main characterization using minimal notions of Pareto-efficiency and consistency, i.e, 3-unanimity and 3-consistency.

In a generalization of our model where objects may remain unassigned, two generalizations of MDPE rules have been characterized. These results rely heavily on a strategic property: strategy-proofness (Ehlers and Klaus [5, 6].) In Section 4 (Corollary 2) we investigate the restrictions imposed on MDPEwt rules by strategy-proofness. In Section 5 we clarify the relation of our results to previous literature.

The characterization of rules in terms of minimal notions of Pareto-efficiency and consistency introduces nontrivial technical challenges (2-neutrality and stategy-proofness are strong relational properties.) We overcome these technical difficulties following a novel "computational" approach that gives an organized answer to the question: if a 2-unanimous and 2-consistent rule chooses some specific allocations, what other allocations must it choose? We believe this method will be useful in understanding the implications of Pareto-efficiency and consistency in other classes of problems (see also the discussion after Theorem 1 for a detailed description of our approach.)

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 describes mixed dictator-pairwise-exchange with tail rules. Section 4 presents our results. Section 5 discusses extensions of our model and results. Appendix A formalizes our computational approach. Appendix B contains proofs.

2 The Model

Let $\mathcal{A}$ be a set of "potential" agents, $\mathcal{X}$ be a set of "potential" objects such that $|\mathcal{A}| \leq |\mathcal{X}|$, and $\mathcal{N} \equiv \{N \subseteq \mathcal{A} : 0 < |N| < \infty\}$ be the set of nonempty finite groups of agents. For each $X \subseteq \mathcal{X}$, the set of linear orders on X is $\mathcal{R}(X)$, i.e., the set of *complete*, *antisymmetric*, and *transitive* binary relations on X.[1] For each $N \in \mathcal{N}$, the set of possible profiles for N on X is $\mathcal{R}(X)^N$. The generic profile is denoted $R = (R_i)_{i \in N} \in \mathcal{R}(X)^N$. For each $N' \subseteq N$, the restriction of $R \in \mathcal{R}(X)^N$ to N' is the profile $R_{N'} \equiv (R_i)_{i \in N'} \in \mathcal{R}(X)^{N'}$. Let $R \in \mathcal{R}(X)^N$, $i \in N$, and $R'_i \in \mathcal{R}(X)$. The profile $(R'_i, R_{-i}) \in \mathcal{R}(X)^N$ is obtained by replacing R_i by R'_i in R.

Let $N \in \mathcal{N}$. An **economy** with agent set N is a pair $e = (X, R)$, where $X \subseteq \mathcal{X}$ is such that $|X| = |N|$, and $R \in \mathcal{R}(X)^N$. The **set of economies with agent set $\boldsymbol{N}$** is $\mathcal{E}^N$ (for each $e = (X, R) \in \mathcal{E}^N$ and each $i \in N$, R_i is agent i's preference relation on X.) We omit the first component in the representation of economies when explicit preferences are available. For example, if $X = \{x, y\}$ and $R = (x\, P_i\, y, x\, P_j\, y)$, we write $e = (x\, P_i\, y, x\, P_j\, y)$.[2] Let $e = (X, R) \in \mathcal{E}^N$. An **allocation for $\boldsymbol{e}$** is a bijection from N into X. The generic allocation is denoted μ. Agent i's allotment under μ is μ_i. The **set of allocations for $\boldsymbol{e}$** is $Z(e)$.

Let $N \in \mathcal{N}$ and $e \in \mathcal{E}^N$. An allocation for e, $\mu \in Z(e)$, is **unanimous for $\boldsymbol{e}$** if there is no $\mu' \in Z(e)$ such that for at least one $j \in N$, $\mu'_j\, P_j\, \mu_j$; it is **Pareto-efficient for $\boldsymbol{e}$** if there is no $\mu' \in Z(e)$ such that for each $i \in N$, $\mu'_i\, R_i\, \mu_i$ and for at least one $j \in N$, $\mu'_j\, P_j\, \mu_j$. The **sets of unanimous** and **Pareto-efficient allocations for $\boldsymbol{e}$** are $U(e)$ and $P(e)$, respectively.[3]

Let $\mathcal{E} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{E}^N$. A **rule** is a function that associates with each economy $e \in \mathcal{E}$ an allocation in $Z(e)$. The generic rule is denoted φ. A rule φ is **unanimous** if for each $e \in \mathcal{E}$ such that $U(e) \neq \emptyset$, $\varphi(e) \in U(e)$; it is **Pareto-efficient** if for each $e \in \mathcal{E}$, $\varphi(e) \in P(e)$. Let $n \in \mathbb{N}$. A rule φ is **$\boldsymbol{n}$-unanimous** if for each $N \in \mathcal{N}$ such that $|N| \leq n$, and each $e \in \mathcal{E}^N$ such that $U(e) \neq \emptyset$, $\varphi(e) \in U(e)$.

Let $N \in \mathcal{N}$, $e \in \mathcal{E}^N$, and $\mu \in Z(e)$. For each $N' \in \mathcal{N}$ such that $N' \subseteq N$, the **reduced economy of $\boldsymbol{e}$ with respect to $\boldsymbol{N'}$ at $\boldsymbol{\mu}$** is $r^{\mu}_{N'}(e) \equiv \left(\bigcup_{i \in N'} \mu_i, R_{N'} | \bigcup_{i \in N'} \mu_i\right)$, where $R_{N'}|A$ is the restriction of $R_{N'}$ to the set A.[4] A rule φ is **consistent** if for each pair $\{N, N'\} \subseteq \mathcal{N}$ such that $N' \subseteq N$, and each $e \in \mathcal{E}^N$, $\varphi(r^{\mu}_{N'}(e)) = \varphi(e)|_{N'}$. Let $n \in \mathbb{N}$.

[1] A binary relation B on X is *antisymmetric* if for each $\{x, y\} \subseteq X$, $x\, B\, y$ and $y\, B\, x$ implies $x = y$.

[2] For each $k \in N$ and each $R_k \in \mathcal{R}(X)$, P_k denotes the asymmetric part of R_k.

[3] If $U(e) \neq \emptyset$, then $|U(e)| = 1$.

[4] Our domain is closed under the reduction operation, i.e, for each pair $\{N, N'\} \subseteq \mathcal{N}$ such that $N' \subseteq N$, each $e \in \mathcal{E}^N$, and each $\mu \in Z(e)$, $r^{\mu}_{N'}(e) \in \mathcal{E}$.

A rule φ is ***n*-consistent** if for each pair $\{N, N'\} \subseteq \mathcal{N}$ such that $N' \subseteq N$ and $|N'| \leq n$, and each $e \in \mathcal{E}^N$, $\varphi(r^{\mu}_{N'}(e)) = \varphi(e)|_{N'}$.

Let $N \in \mathcal{N}$, $\{X, X'\}$ be a pair of subsets of $\mathcal{X}$ such that $|X| = |X'| = |N|$, and $\sigma : X \to X'$ be a bijection. Let $R \in \mathcal{R}(X)^N$. The **relabeling of $\boldsymbol{R}$ with respect to $\boldsymbol{\sigma}$** is the profile $R^{\sigma} \equiv (R^{\sigma}_i)_{i \in N} \in \mathcal{R}(X')^N$ such that, for each pair $\{x, y\} \subseteq X'$ and each $i \in N$, $x\, R^{\sigma}_i\, y$ if and only if $\sigma^{-1}(x)\, R_i\, \sigma^{-1}(y)$. Let $n \in \mathbb{N}$. A rule φ is ***n*-neutral** if for each $N \in \mathcal{N}$ such that $|N| \leq n$, each $e = (X, R) \in \mathcal{E}^N$, each $X' \subseteq \mathcal{X}$ such that $|X'| = |N|$, and each bijection $\sigma : X \to X'$, $\varphi(X', R^{\sigma}) = \sigma \circ \varphi(e)$.

A rule φ is **strategy-proof** if for each $N \in \mathcal{N}$, each $e = (X, R) \in \mathcal{E}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}(X)$, $\varphi_i(e)\, R_i\, \varphi_i(X, (R'_i, R_{-i}))$.

3 Mixed dictator-pairwise-exchange with tail rules

In this section we describe three families of *Pareto-efficient* and *consistent* rules. Subsection 3.1 describes mixed dictator-pairwise-exchange rules (Ehlers [7].) Subsection 3.2 describes a family of rules when the set of potential agents has cardinality, at most, three: "tail rules". Subsection 3.3 describes the family of rules that we characterize in our main theorem: "mixed dictator-pairwise-exchange with tail rules".

3.1 Mixed dictator-pairwise-exchange rules

Let I be an index set and $\Pi = (\pi_k)_{k \in I}$ be a partition of $\mathcal{A}$ such that for each $k \in I$, $|\pi_k| \leq 2$.[5] Let $\succ$ be a linear order on Π. The order $\succ$ can be interpreted as a social priority on Π. Let $\Pi_2 \equiv \{\pi \in \Pi : |\pi| = 2\}$ be the **set of pairs in $\boldsymbol{\Pi}$**. A **tie-breaker on $\boldsymbol{\Pi_2}$** is a function $T : \Pi_2 \times \mathcal{X} \to \mathcal{A}$ such that for each $\pi \in \Pi_2$ and each $x \in \mathcal{X}$, $T(\pi, x) \in \pi$ and $T(\pi, \mathcal{X}) = \pi$.[6] We call a triple $(\Pi, \succ, T)$ a **mixed dictator-pairwise-exchange (MDPE) structure**.

The **MDPE rule associated to $\boldsymbol{(\Pi, \succ, T)}$**, denoted $D^{(\Pi, \succ, T)}$, is defined through the following algorithm. Let $N \in \mathcal{N}$ and $e = (X, R) \in \mathcal{E}^N$. Let Π^N be the partition of N induced by Π, and $\succ^N$ be the linear order on Π^N induced by $\succ$.[7] Let π^N_1 be the maximal set in Π^N with respect to (w.r.t.) $\succ^N$. If π^N_1 is a singleton, the agent in that set receives her preferred object in X. If π^N_1 is a pair, both agents in π^N_1 receive their preferred objects if possible. If their preferred objects are the same, say $x \in X$, the agent favored by the tie-breaker, $T(\pi^N_1, x) \in \pi^N_1$, receives x. The other agent receives

[5] $\bigcup_I \pi_k = \mathcal{N}$; for each $k \in I$, $\pi_k \neq \emptyset$; and for each $\{k, k'\} \subseteq I$ such that $k \neq k'$, $\pi_k \cap \pi_{k'} = \emptyset$.

[6] If $\Pi_2 = \emptyset$, each tie-breaker on Π_2 is trivial; we denote it $T^{\emptyset}$.

[7] $\Pi^N \equiv \{\pi^N = \pi \cap N \neq \emptyset : \pi \in \Pi\}$.

her preferred object in $X \setminus \{x\}$. The remaining objects are allocated analogously among the remaining agents. Let π_2^N be the maximal set in $\Pi^N \setminus \pi_1^N$ w.r.t. $\succ^N$. If π_2^N is a singleton, the agent in that set receives her preferred object in $X \setminus \{D_i^{(\Pi,\succ,T)}(e) : i \in \pi_1^N\}$, and so on.

MDPE rules are equivalent to the restricted endowment inheritance rules (Ehlers, Klaus, and Pápai [3]), and a subclass of the hierarchical exchange rules (Pápai [11]; and Ehlers, Klaus, and Pápai [3].) Let $(\Pi, \succ, T)$ be an MDPE structure. If Π_2 is empty, then $D^{(\Pi,\succ,T^\emptyset)}$ is called a **serial dictatorship** and denoted $D^{\succ}$. If Π_2 contains, at most, the maximal set of Π w.r.t. $\succ$, then $D^{(\Pi,\succ,T)}$ is called a **bi-polar dictatorship** (Bogomolnaia, Deb, and Ehlers [2].)

Example 1. *An MDPE rule. Let* $\mathcal{A} = \{1, 2, 3, 4\}$, $\mathcal{X} = \{a, b, c, d\}$, $(\Pi, \succ) \equiv \{1\} \succ \{2, 3\} \succ \{4\}$, *and* $T(\{2, 3\}, x) \equiv 2$ *if and only if* $x \in \{a, b\}$. *We illustrate the recommendations given by* $D^{(\Pi,\succ,T)}$ *in four sample economies. Allocations are depicted between brackets.*

R_1	R_2	R_4
$[a]$	a	a
b	$[b]$	b
c	c	$[c]$

R_1	R_2	R_3	R_4
$[a]$	a	a	a
b	$[b]$	$[d]$	d
c	c	c	$[c]$
d	d	b	b

R_1	R_2	R_3	R_4
$[a]$	c	a	a
b	a	$[c]$	$[d]$
c	$[b]$	d	c
d	d	b	b

R_1	R_2	R_3	R_4
$[a]$	$[b]$	a	a
b	a	b	d
c	c	$[d]$	$[c]$
d	d	c	b

3.2 Tail Rules

There are *Pareto-efficient* and *consistent* rules that are not MDPE rules. To see this, suppose that there are two potential agents; then, each *Pareto-efficient* rule is *consistent.* In this subsection we describe a family of *Pareto-efficient* and *consistent* rules when there are at most three potential agents. In the next section we use these rules to describe the family of rules that we characterize in our main theorem.

We first define the rules when $|\mathcal{A}| = 3$. Let $\gamma : \{1, 2, 3\} \to \mathcal{A}$ be a bijection, $y^* \in \mathcal{X}$, and $Y \subseteq \mathcal{X} \setminus \{y^*\}$ be a nonempty set of objects. Let $\mathcal{E}_{\gamma,y^*,Y} \subseteq \mathcal{E}^{\mathcal{N}}$ be the set:[8]

[8] A typical economy in $\mathcal{E}_{\gamma,y^*,Y}$ is

$R_{\gamma(1)}$	$R_{\gamma(2)}$	$R_{\gamma(3)}$
y^*	z	z
x	y^*	$\{y^*, x\}$
z	x	

, where $z \in Y$.

$$\mathcal{E}_{\gamma, y^*, Y} \equiv \left\{ (X,R) \in \mathcal{E}^{\mathcal{N}} : \begin{array}{l} X = \{y^*, z, x\}\ , z \in Y,\ y^* \, P_{\gamma(1)} \, x \, P_{\gamma(1)} \, z, \\ z \, P_{\gamma(2)} \, y^* \, P_{\gamma(2)} \, x,\ z \, P_{\gamma(3)} \, y^*,\ z \, P_{\gamma(3)} \, x. \end{array} \right\}.$$

Let $E \subseteq \mathcal{E}_{\gamma,y^*,Y}$. We call a list (γ, y^*, Y, E) a **tail structure**. The **tail rule associated to $(\boldsymbol{\gamma}, \boldsymbol{y^*}, \boldsymbol{Y}, \boldsymbol{E})$**, denoted $\mathcal{T}^{(\gamma,y^*,Y,E)}$, coincides with $D^{\gamma(1)\succ\gamma(2)\succ\gamma(3)}$, except in the five cases listed in Table 1.

Case	$N \in \mathcal{N}$	$X \subseteq \mathcal{X}$	$R \in \mathcal{R}(X)^N$	$D^{\succ}$
I	$\{\gamma(2), \gamma(3)\}$	$\{y^*, z\}, z \in Y$	$z \, P_{\{\gamma(2),\gamma(3)\}} \, y^*$	$\gamma(3) \succ \gamma(2)$
II	$\{\gamma(1), \gamma(2)\}$	$y^* \in X$	$y^* \, P_{\{\gamma(1),\gamma(2)\}} \, X \setminus y^*$	$\gamma(2) \succ \gamma(1)$
III	$\mathcal{A}$	$y^* \in X$	$y^* \, P_{\{\gamma(1),\gamma(2)\}} \, X \setminus \{y^*\}$	$\gamma(2) \succ \gamma(1) \succ \gamma(3)$
IV	$\mathcal{A}$	$\{x, y^*, z\},\ z \in Y$	$x \, P_{\gamma(1)} \, X \setminus \{x\}, z \, P_{\gamma(3)} \, y^*$	$\gamma(1) \succ \gamma(3) \succ \gamma(2)$
V	$e \in E$			$\gamma(3) \succ \gamma(2) \succ \gamma(1)$

Table 1: Exceptions in the definition of the tail rule $\boldsymbol{\mathcal{T}^{(\gamma,y^*,Y,E)}}$. Let $N \in \mathcal{N}$ and $e = (X,R) \in \mathcal{E}^N$; e falls into Cases I to IV if N is the set listed in Column 2 and X and R satisfy the requirements in Columns 3 and 4; e falls into Case V if $e \in E$. In each case, $\mathcal{T}^{(\gamma,y^*,Y,E)}(e) = D^{\succ}(e)$, where the order $\succ$ is given by the respective entry in Column 5.

Let (γ, y^*, Y, E) be a tail structure. The rule $\mathcal{T}^{(\gamma,y^*,Y,E)}$ coincides with $D^{\gamma(1)\succ\gamma(2)\succ\gamma(3)}$ in each two-agent economy except in two cases (I and II in Table 1.) For each economy $e \in \mathcal{E} \setminus \mathcal{E}_{\gamma,y^*,Y}$ there is a unique allocation in $P(e)$ that is "consistent" with the choices of $\mathcal{T}^{(\gamma,y^*,Y,E)}$ in two-agent economies; this allocation is $D^{\gamma(1)\succ\gamma(2)\succ\gamma(3)}(e)$ except in two cases (III and IV in Table 1). For each economy $e \in \mathcal{E}_{\gamma,y^*,Y}$ there are two allocations in $P(e)$ that are "consistent" with the choices of $\mathcal{T}^{(\gamma,y^*,Y,E)}$ in two-agent economies: $D^{\gamma(1)\succ\gamma(2)\succ\gamma(3)}(e)$ and $D^{\gamma(3)\succ\gamma(1)\succ\gamma(2)}(e)$. The set E contains all the economies in $\mathcal{E}_{\gamma,y^*,Y}$ for wich $\mathcal{T}^{(\gamma,y^*,Y,E)}$ chooses $D^{\gamma(3)\succ\gamma(1)\succ\gamma(2)}(e)$.

Now, let $\mathcal{A}$ be such that $|\mathcal{A}| \leq 3$. A rule φ is a **tail rule** if one of the following three conditions holds: (1) $|\mathcal{A}| = 1$, (2) $|\mathcal{A}| = 2$, φ is *Pareto-efficient*, but it is not a serial dictatorship, or (3) $|\mathcal{A}| = 3$ and there exists a tail structure, (γ, y^*, Y, E), such that $\varphi = \mathcal{T}^{(\gamma,y^*,Y,E)}$.

Remark 1. *Tail rules are Pareto-efficient and consistent.*

The proof is in Appendix B.

Example 2. *A tail rule when $|\mathcal{A}| = 2$ that is not a MDPE rule. Let $\mathcal{A} = \{1, 2\}$ and $\mathcal{X} = \{a, b, c, d\}$. Let φ be the Pareto-efficient rule that chooses the allocations listed* *in the following table (there is a unique Pareto-eficient allocation for each of the other economies.)*

R_1	R_2
$[a]$	a
b	$[b]$

R_1	R_2
a	$[a]$
$[c]$	c

R_1	R_2
a	$[a]$
$[d]$	d

R_1	R_2
$[b]$	b
a	$[a]$

R_1	R_2
b	$[b]$
$[c]$	c

R_1	R_2
$[b]$	b
d	$[d]$

R_1	R_2
c	$[c]$
$[a]$	a

R_1	R_2
$[c]$	c
b	$[b]$

R_1	R_2
c	$[c]$
$[d]$	d

R_1	R_2
d	$[d]$
$[a]$	a

R_1	R_2
d	$[d]$
$[b]$	b

R_1	R_2
$[d]$	d
c	$[c]$

Observe that φ is not an MDPE rule. None of the agents has priority over the other agent. Besides, it is not induced by a tie-breaker. In the first two economies, agents 1 and 2 rank a above the other object available, but the agent who receives a is not the same.

Example 3. *A tail rule when $|\mathcal{A}| = 3$. Let $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{X} = \{a, b, c, d\}$. Let γ be such that for each $i \in \mathcal{A}$, $\gamma(i) = i$. Let $y^* = a$, $Y = \{b, c\}$, and $E = \{e_1, e_2\}$, where $e_1 = (a\, P_1\, b\, P_1\, c, c\, P_2\, a\, P_2\, b, c\, P_3\, a\, P_3\, b)$ and $e_2 = (a\, P_1\, d\, P_1\, b, b\, P_2\, a\, P_2\, d, b\, P_3\, d\, P_3\, a)$. We first illustrate the recommendations given by $\mathcal{T}^{(\gamma,a,Y,E)}$ in two-agent economies:*

R_1	R_2
$[b]$	b
c	$[c]$

R_1	R_3
$[b]$	b
a	$[a]$

R_2	R_3
$[b]$	b
c	$[c]$

R_2	R_3
$[d]$	d
a	$[a]$

R_2	R_3
b	$[b]$
$[a]$	a

R_1	R_2
a	$[a]$
$[b]$	b

From left to right, the first four economies do not fall into the exceptional cases in Table 1. The last two economies fall into Cases I and II, respectively. Now, we illustrate the recommendations given by $\mathcal{T}^{(\gamma,a,Y,E)}$ in three-agent economies:

R_1	R_2	R_3
$[b]$	b	b
a	$[a]$	a
c	c	$[c]$

R_1	R_2	R_3
$[a]$	$[b]$	b
d	a	a
b	d	$[d]$

R_1	R_2	R_3
a	$[a]$	a
$[b]$	c	$[c]$
c	b	b

R_1	R_2	R_3
$[b]$	c	$[c]$
a	$[a]$	b
c	b	a

R_1	R_2	R_3
a	b	$[b]$
$[d]$	$[a]$	d
b	d	a

From left to right, the first two economies do not fall into the exceptional cases in Table 1. The last three economies fall into Cases III, IV, and V, respectively.

3.3 Mixed dictator-pairwise-exchange with tail rules

We turn now to the definition of a broader family of rules that generalizes MDPE and tail rules. We describe them using an analogous algorithm to the one used to describe MDPE rules.

Let I be an index set and $\Pi = (\pi_k)_{k \in I}$ be a partition of $\mathcal{A}$ such that for each $k \in I$, $|\pi_k| \leq 3$ and $|\{k \in I : |\pi_k| = 3\}| \leq 1$. That is, Π is a partition of $\mathcal{A}$ in sets of, at most, three agents with the further restriction that, at most, one set has three agents. Let $\succ$ be a linear order on the partition Π such that, if $|\pi_k| = 3$, then for each $k' \neq k$, $\pi_{k'} \succ \pi_k$. The order $\succ$ can be interpreted as a social priority on Π. If there is a set $\pi_k \in \Pi$ such that for each $k' \in I$ such that $k' \neq k$, $\pi_{k'} \succ \pi_k$, we call such a set the **tail of Π w.r.t. $\succ$**. We denote it $\tau(\Pi, \succ)$.[9] Observe that if there is a set of three agents in Π, it is $\tau(\Pi, \succ)$.

Let $\Pi_2 \backslash \tau(\Pi, \succ) \equiv \{\pi \in \Pi \backslash \{\tau(\Pi, \succ)\} : |\pi| = 2\}$ be the **set of pairs in $\Pi \setminus \{\tau(\Pi, \succ)\}$**. A **tie-breaker on $\Pi_2 \setminus \tau(\Pi, \succ)$** is a function $T : \Pi_2 \setminus \tau(\Pi, \succ\ ) \times \mathcal{X} \to \mathcal{A}$, such that for each $\pi \in \Pi_2 \setminus \tau(\Pi, \succ)$ and each $x \in \mathcal{X}$, $T(\pi, x) \in \pi$ and $T(\pi, \mathcal{X}) = \pi$.[10] Let Q be a tail rule defined on the subdomain of economies with set of potential agents $\tau(\Pi, \succ)$ and set of potential objects $\mathcal{X}$.[11] We call a list $(\Pi, \succ, T, Q)$ a **mixed dictator-pairwise-exchange structure with tail**.

The **mixed dictator-pairwise-exchange with tail (MDPEwt) rule associated to $(\Pi, \succ, T, Q)$**, denoted $D^{(\Pi, \succ, T, Q)}$, is defined through the following algorithm. Let $N \in \mathcal{N}$ and $e = (X, R) \in \mathcal{E}^N$. Let Π^N be the partition induced by Π on N, and $\succ^N$ be the linear order induced by $\succ$ on Π^N. Let π_1^N be the maximal set in Π_N w.r.t. $\succ^N$. Suppose $\pi_1^N \neq \tau(\Pi, \succ) \cap N$. If π_1^N is a singleton, the agent in that set receives her preferred object in X. If π_1^N is a pair, both agents in π_1^N receive their preferred object if possible. If their preferred objects are the same, say $x \in X$, the agent favored by the tie-breaker, $T(\pi_1^N, x) \in \pi_1^N$, receives x. The other agent receives her preferred object in $X \setminus \{x\}$. The remaining objects are allocated analogously among the remaining groups in $\Pi^N \setminus \tau(\Pi, \succ)$. Let π_2^N be the maximal set in $\Pi^N \setminus \pi_1^N$ w.r.t. $\succ^N$. Suppose $\pi_2^N \neq \tau(\Pi, \succ) \cap N$. If π_2^N is a singleton, the agent in that set receives her preferred object in $X \setminus \{D_i^{(\Pi, \succ, T, Q)}(e) : i \in \pi_1^N\}$, and so on. The agents in $\tau(\Pi, \succ) \cap N$ receive the objects recommended by Q in economy $(X', R_{\tau(\Pi, \succ)}|X') \in \mathcal{E}^{\tau(\Pi, \succ) \cap N}$, where $X' = X \setminus \{D_i^{(\Pi, \succ, T, Q)}(e) : i \in N \setminus \tau(\Pi, \succ)\}$.

MDPEwt rules generalize MDPE and tail rules. The algorithm that defines a MDPEwt rule, $D^{(\Pi, \succ, T, Q)}$, is equivalent to the one defining an MDPE rule up to the last component of Π. The allocation for agents in this last component is determined by the tail rule Q.

Remark 2. *MDPEwt rules are Pareto-efficient and consistent.*

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[9] If for each $\pi \in \Pi$, there is $\pi' \in \Pi$ such that $\pi \succ \pi'$, let $\tau(\Pi, \succ) \equiv \emptyset$.

[10] If $\Pi_2 \setminus \tau(\Pi, \succ) = \emptyset$, each tie-breaker on $\Pi_2 \setminus \tau(\Pi, \succ)$ is trivial; we denote it $T^\emptyset$.

[11] If $|\tau(\Pi, \succ)| \leq 1$, each tail rule Q is trivial; we denote it $Q^\emptyset$.

The proof is in Appendix B.

Example 4. *A MDPEwt rule whose tail has cardinality 1. Let* $\mathcal{A} = \{1,2,3,4\}$ *and* $\mathcal{X} = \{a,b,c,d\}$. *Let* $(\Pi,\succ) \equiv \{1\} \succ \{2,3\} \succ \{4\}$, *and* $T(\{2,3\},x) \equiv 2$ *if and only if* $x \in \{a,b\}$. *Observe that* $D^{(\Pi,\succ,T,Q^{\emptyset})} \equiv D^{(\Pi,\succ,T)}$, *where* $D^{(\Pi,\succ,T)}$ *was defined in Example 1.*

Example 5. *A MDPEwt rule whose tail has cardinality 2. Let* $\mathcal{A} = \{1,2,3,4\}$ *and* $\mathcal{X} = \{a,b,c,d\}$. *Let* $(\Pi,\succ) \equiv \{3,4\} \succ \{1,2\}$, *and* $T(\{3,4\},x) \equiv 3$ *if and only if* $x \in \{b,c\}$. *Let* Q *be the tail rule defined in Example 2. We illustrate* $D^{(\Pi,\succ,T,Q)}$ *by applying it to three sample economies:*

R_4	R_1	R_2
$[a]$	a	a
b	b	$[b]$
c	$[c]$	c

R_3	R_4	R_1
$[b]$	b	b
a	$[a]$	a
c	c	$[c]$

R_3	R_4	R_1	R_2
a	$[a]$	a	a
$[b]$	b	$[d]$	d
c	c	c	$[c]$
d	d	b	b

Observe that since the tail rule Q *is not an MDPE rule, then neither is the rule* $D^{(\Pi,\succ,T,Q)}$.

Example 6. *A MDPEwt rule whose tail has cardinality 3. Let* $\mathcal{A} = \{1,2,3,4\}$, $\mathcal{X} = \{a,b,c,d\}$, *and* $(\Pi,\succ) \equiv \{4\} \succ \{1,2,3\}$. *Let* Q *be the tail rule defined in Example 3. Observe that for each* $e \in \bigcup_{N\subseteq\{1,2,3\}} \mathcal{E}^N$, $D^{(\Pi,\succ,T^{\emptyset},Q)}(e) = Q(e)$. *We illustrate* $D^{(\Pi,\succ,T^{\emptyset},Q)}$ *by applying it to sample economies for groups of agents containing agent 4:*

R_4	R_1
$[a]$	a
b	$[b]$

R_4	R_1	R_2
$[a]$	a	a
b	$[b]$	b
c	c	$[c]$

R_4	R_1	R_3
$[b]$	$[a]$	a
a	b	$[b]$
c	c	c

R_4	R_1	R_2
$[b]$	a	$[a]$
b	b	b
c	$[c]$	c

R_4	R_1	R_2	R_3
$[d]$	a	$[a]$	a
a	$[b]$	c	$[c]$
b	c	b	b
c	d	d	d

4 Results

Our main theorem characterizes MDPEwt rules in terms of minimal notions of *Pareto-efficiency* and *consistency.*

Theorem 1. *Assume* $|\mathcal{X}| \geq 4$. *A rule is 3-unanimous and 3-consistent if and only if it is an MDPEwt rule.*

The proof is in Appendix B. We first give the intuition of the proof that if a rule satisfies *3-unanimity* and *3-consistency*, then it is an MDPEwt rule (the other direction is implied by Remark 2.) Given a rule φ satisfying *2-unanimity* and *2-consistency*, we identify the restrictions on the allocations chosen by φ in two-agent economies that are implied by the properties satisfied by the rule. Using these restrictions, we identify an MDPE structure with tail, $(\Pi, \succ, T, Q)$, such that if φ satisfies *3-unanimity* and *3-consistency*, then $\varphi = D^{(\Pi,\succ,T,Q)}$.

Our proof is a classical "calibration-verification" procedure. The novelty of our approach is the way in which we obtain the restrictions induced by *2-unanimity* and *2-consistency* on the allocations in two-agent economies. We sketch our approach. Consider a rule φ. The recommendations made by φ in two-agent economies in wich both agents have the same preferences induce a family of binary relations on the set of potential agents. This family is indexed with ordered pairs of objects. Thus, the family of binary relations induced by a rule is a subset of the cross product $\mathcal{X}^2 \times \mathcal{A}^2$.

Suppose that φ is *2-unanimous* and *2-consistent*, and that its induced family of binary relations contains a set $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$. Let $N \in \mathcal{N}$, $e \in \mathcal{E}^N$, and $\mu \in Z(e)$. If there is a pair $\{i, j\} \subseteq N$ such that $\mu|_{\{i,j\}}$ is not "compatible" with the set h, then we can say $\varphi(e) \neq \mu$. Possibly, there are economies $e \in \mathcal{E}$ such that just one allocation in $Z(e)$ survives this test. In such a case, we know exactly $\varphi(e)$.[12] Now, since φ is *2-consistent*, then for each pair $\{i, j\} \subseteq N$, we know $\varphi(e)|_{\{i,j\}}$. This construction may give us new points that should belong to the family of binary relations induced by φ. These new points are "implications" of h obtained by analyzing economy e. In this way, we identify the set of "implications" of h that are deduced from the analysis of economy e, $\mathcal{I}(h, e)$ (Subsection 4.1 and Lemma 4 in Appendix A.)

Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $E \subseteq \mathcal{E}$ be a finite set of economies. Our next step is to identify the set of implications of h that are deduced from the analysis of the set of economies E, $\mathcal{I}(h, E)$ (Algorithm 1, Appendix A.) Intuitively, from the "recursive" analysis of the set of economies E, we know that the set $\mathcal{I}(h, E)$ must be contained in each of the family of binary relations that contains the set h and is induced by a *2-unanimous* and *2-consistent* rule.

Using this "implications operator" we find "independence and transitivity properties" of the binary relations induced by *2-unanimous* and *2-consistent* rules (Lemmas 1, 2, and 3; see also the proof of Theorem 1 for $|\mathcal{A}| \leq 3$.) These properties translate directly as restrictions on the recommendations made by such rules in two-agent economies.

[12] This construction is known as the "Bracing lemma" (Thomson [13].) See also Lemma 1 for a detailed example.

Theorem 1 is tight. If $|\mathcal{X}| = 3$ and $|\mathcal{A}| = 3$, there is a *Pareto-efficient* and *consistent* rule that is not an MDPEwt rule (Appendix B, Example 8.)[13] If $|\mathcal{X}| \geq 4$ and $|\mathcal{A}| \geq 3$, there is a *2-unanimous* and *consistent* rule that is not *3-unanimous*; in particular it is not an MDPEwt rule (Appendix B, Example 9.) If $|\mathcal{X}| \geq 4$ and $|\mathcal{A}| \geq 4$, there is a *Pareto-efficient* and *2-consistent* rule that is not *3-consistent*; in particular it is not a MDPEwt rule (Appendix B, Example 10.)

If a rule is *unanimous* and *consistent*, then it is *Pareto-efficient* (Takamiya [12].) The following corollary of Theorem 1 and Remark 2 generalizes this result.

Corollary 1. *Assume $|\mathcal{X}| \geq 4$. If a rule is 3-unanimous and 3-consistent, then it is Pareto-efficient and consistent.*

MDPE rules are *strategy-proof* (Ehlers [7].) However, there are tail rules that are not. Thus, there are MDPEwt rules that are not *strategy-proof*. The following proposition characterizes the class of tail rules that are.

Proposition 1. *Assume $|\mathcal{A}| = 3$. A tail rule $\mathcal{T}^{(\gamma, y^*, Y, E)}$ is strategy-proof if and only if $E = \emptyset$.*

The proof is in Appendix B. This characterization of *strategy-proof* tail rules sheds additional light on the structure of tail rules. In particular, observe that the recommendations made by *strategy-proof* tail rules can be calculated with "endowment inheritance tables" (Pápai [11].)[14]

The following result is a straightforward consequence of Proposition 1. It characterizes the set of *strategy-proof* MDPEwt rules.

Corollary 2. *A MDPEwt rule $D^{(\Pi, \succ, T, Q)}$ is strategy-proof if and only if $|\tau(\Pi, \succ)| \leq 2$ or there is a tail structure $(\gamma, y^*, Y, \emptyset)$ such that $Q = \mathcal{T}^{(\gamma, y^*, Y, \emptyset)}$.*

[13] Theorem 1 holds if $|\mathcal{A}| \leq 2$ for arbitrary cardinality of $\mathcal{X}$.

[14] For each three-agent economy $e = (X, R) \in \mathcal{E}^{\mathcal{A}}$, $\mathcal{T}^{(\gamma, y^*, Y, \emptyset)}(e)$ can be calculated using the following "endowment inheritance tables",

$X \cap Y$	$X \cap Y^c \setminus \{y^*\}$	y^*
$\gamma(1)$	$\gamma(1)$	$\gamma(2)$
$\gamma(3)$	$\gamma(2)$	$\gamma(1)$
$\gamma(2)$	$\gamma(3)$	$\gamma(3)$

if $y^* \in X$, and

X
$\gamma(1)$
$\gamma(2)$
$\gamma(3)$

otherwise.

4.1 Restrictions on the recommendations in two-agent economies for a 2-unanimous and 2-consistent rule

Let φ be a rule and $F^{\varphi} \equiv \left\{\succsim^{\varphi}_{xy}: x, y \in \mathcal{X}, x \neq y\right\} \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ be the family of binary relations on $\mathcal{A}$ such that, for each $\{i,j\} \subseteq \mathcal{A}$ and each $\{x,y\} \subseteq \mathcal{X}$ such that $x \neq y$, $i \succ^{\varphi}_{xy} j$ if and only if $\varphi_i(xP_iy, xP_jy) = x$. In words, $i \succ^{\varphi}_{xy} j$ if φ gives priority to agent i over agent j in economy (xP_iy, xP_jy). We observe that for each rule φ, F^{φ} is a family of *antisymmetric* and *complete* relations on $\mathcal{A}$.

Let $F^{\varphi}_{\mathcal{X}} \equiv \{\succsim^{\varphi}_{x}: x \in \mathcal{X}\}$ be the family of binary relations on $\mathcal{A}$ such that, for each $\{i,j\} \subseteq \mathcal{A}$ and each $x \in \mathcal{X}$, $i \succ^{\varphi}_{x} j$ if and only if for each $y \in \mathcal{X}\setminus\{x\}$, $i \succ^{\varphi}_{xy} j$. In words, $i \succ^{\varphi}_{x} j$ if φ gives priority to agent i over agent j in economy $(x\,P_i\,y, x\,P_j\,y)$, independently of y. We observe that for each rule φ, $F^{\varphi}_{\mathcal{X}}$ is a family of *antisymmetric*, but not necessarily *complete*, relations on $\mathcal{A}$.

Let $\succsim^{\varphi}$ be the binary relation on $\mathcal{A}$ such that, for each $\{i,j\} \subseteq \mathcal{A}$, $i \succ^{\varphi} j$ if and only if for each $x \in \mathcal{X}$, $i \succ^{\varphi}_{x} j$. In words, $i \succ^{\varphi} j$ if the rule φ gives priority to agent i over agent j in every two-agent economy. We observe that for each φ, the relation $\succsim^{\varphi}$ is *antisymmetric*, but not necessarily *complete*.

Lemma 1. *Let φ be a 2-unanimous and 2-consistent rule, and $\{i,j,k\} \subseteq \mathcal{A}$. If for some $\{x,y\} \subseteq \mathcal{X}$ such that $x \neq y$, $i \succ^{\varphi}_{xy} j$ and $j \succ^{\varphi}_{xy} k$, then for each $z \in \mathcal{X}\setminus\{x,y\}$, $i \succ^{\varphi}_{zy} k$.*[15]

Proof. Let $z \in \mathcal{X}\setminus\{x,y\}$. We want to prove that if $i \succ^{\varphi}_{xy} j$ and $j \succ^{\varphi}_{xy} k$, then $i \succ^{\varphi}_{zy} k$. Suppose $i \succ^{\varphi}_{xy} j$ and $j \succ^{\varphi}_{xy} k$, and consider the three-agent economy $e = (z\,P_i\,x\,P_i\,y, x\,P_j\,y\,P_j\,z, x\,P_k\,z\,P_k\,y)$. We claim that $\varphi_i(e) = z$, $\varphi_j(e) = x$, and $\varphi_k(e) = y$. Table 2 summarizes the reasons why the other allocations for e are ruled out. Consider the reduced economy of e w.r.t. $\{i,k\}$ at $\varphi(e)$, $r^{\varphi(e)}_{\{i,k\}}(e) = (z\,P_i\,y, z\,P_k\,y)$. Using *2-consistency* of φ, $\varphi(z\,P_i\,y, z\,P_k\,y) = \varphi(e)\,|_{\{i,k\}}$. Thus, $i \succ^{\varphi}_{zy} k$. □

A straightforward consequence of Lemma 1 is the following result.

Corollary 3 (Ergin [8])**.** *A rule φ is 2-unanimous, 2-consistent, and 2-neutral if and only if it is a serial dictatorship rule.*

Lemma 2. *Assume $|\mathcal{X}| \geq 4$. Let φ be a 2-unanimous and 2-consistent rule, and $\{i,j,k\} \subseteq \mathcal{A}$. If for some $\{x,y\} \subseteq \mathcal{X}$ such that $x \neq y$, $i \succ^{\varphi}_{xy} j$ and $j \succ^{\varphi}_{xy} k$, then $i \succ^{\varphi}_{x} \{j,k\}$.*

[15] This lemma is an application of the Bracing Lemma (see Thomson [13].)

Agent	i	j	k	Description
Allocation μ	x	y	z	$\mu\|_{\{i,k\}} \notin U(r^{\mu}_{\{i,k\}}(e)) \neq \emptyset$
	x	z	y	$\mu\|_{\{i,j\}} \notin U(r^{\mu}_{\{i,j\}}(e)) \neq \emptyset$
	y	x	z	Contradicts $i \succ^{\varphi}_{xy} j$, i.e., $\mu\|_{\{i,j\}} \neq \varphi\left(r^{\mu}_{\{i,j\}}(e)\right)$.
	y	z	x	$\mu\|_{\{i,j\}} \notin U(r^{\mu}_{\{i,j\}}(e)) \neq \emptyset$
	z	x	y	
	z	y	x	Contradicts $j \succ^{\varphi}_{xy} k$, i.e., $\mu\|_{\{j,k\}} \neq \varphi\left(r^{\mu}_{\{j,k\}}(e)\right)$.

Table 2: Proof of Lemma 1.

The proof is in Appendix B. Observe that in particular, Lemma 2 implies that if a rule φ is *2-unanimous* and *2-consistent*, then if $|\mathcal{X}| \geq 4$, F^{φ} is a family of *transitive* binary relations on $\mathcal{A}$.

Lemma 3. *Assume $|\mathcal{A}| \geq 4$. Let φ be a 2-unanimous and 2-consistent rule, and $\{i,j,k,l\} \subseteq \mathcal{A}$. If for some $\{x,y\} \subseteq \mathcal{X}$ such that $x \neq y$, $i \succ^{\varphi}_{xy} j \succ^{\varphi}_{xy} k \succ^{\varphi}_{xy} l$, then $i \succ^{\varphi} \{k,l\}$.*

The proof is in Appendix B.

5 Discussion

In this section we discuss extensions of our model and results. Two main features of our definition of an economy are: the set of agents and the social endowment have the same cardinality, and preferences are defined on the social endowment (not on the set of potential objects.)

Suppose now that the cardinality of the set of agents and the cardinality of the social endowment are not required to be equal, and that preferences are defined on the set of potential objects and the possibility to receive no object. This model has been studied recently in the literature (Ehlers, Klaus, and Pápai [3]; and Ehlers and Klaus [4, 5, 6].)

If objects may remain unassigned, *reduced economies with respect to a group at an allocation* may be endowed with: (1) the union of the objects the agents in the group were recommended to receive, and (2) the objects in the economy that were not assigned to agents in the complement of the group. These two definitions lead to two different notions of consistency. The notion of consistency corresponding to definition (1) has been named *reallocation-consistency* (Ehlers and Klaus [5].) The notion of consistency corresponding to definition (2) has been named *consistency* (Ehlers and Klaus [6].)

Pareto-efficient, *reallocation-consitent,* and *strategy-proof* rules are characterized as suitable extensions of MDPE rules (Ehlers and Klaus [5].) *Pareto-efficient*, *consistent*, and *strategy-proof* rules are characterized as suitable extensions of *strategy-proof* MDPEwt rules (Ehlers and Klaus [6].)[16] On the one hand, axioms are independent in both characterizations and our results based on *Pareto-efficiency* and *consistency* do not extend to this model. On the other hand, our main characterization is not implied by these theorems. Our restrictions on the domain of economies seen as "independence properties" are weaker than the "independence properties" implied by *strategy-proofness*.

6 Appendix A: computational approach

Let $X = \{x, y, z\} \subseteq \mathcal{X}$. Let R^{xyz} denote the preference $x\, P\, y\, P\, z$. Using this notation, $\mathcal{R}(X) = \{R^{xyz}, R^{xzy}, R^{yxz}, R^{yzx}, R^{zxy}, R^{zyx}\}$.

Let $N \in \mathcal{N}$ and $e = (X, R) \in \mathcal{E}^N$. For each $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$, let $A(h, e) \subseteq Z(e)$ be the set

$$A(h,e) \equiv \left\{ \mu \in Z(e) : \begin{array}{l} \text{for each } \{i,j\} \subseteq N \text{ s.t. } U(r^{\mu}_{\{i,j\}}(e)) \neq \emptyset,\ \mu|_{\{i,j\}} \in U(r^{\mu}_{\{i,j\}}(e)), \\ \text{and } \nexists \{i,j\} \subseteq N \text{ s.t. } \mu_j\, P_{\{i,j\}}\, \mu_i, \text{and } \{i \succ_{\mu_j \mu_i} j\} \subseteq h \end{array} \right\}.$$

The set $A(h, e)$ contains the allocations in $Z(e)$ such that for each rule φ with $h \subseteq F^{\varphi}$, if $\varphi(e) \notin A(h, e)$, then φ is neither *2-unanimous* nor *2-consistent*.[17] For example, from the proof of Lemma 1, it is clear that the unique allocation μ in $A(h, e)$ where $h = \{i \succ_{xy} j,\, j \succ_{xy} k\}$ and $e = \left(R_i^{zxy}, R_j^{xyz}, R_k^{xzy}\right)$, is $\mu_i = z$, $\mu_j = x$, and $\mu_k = y$. That is, if φ is such that $h \subseteq F^{\varphi}$, then if $\varphi(e) \notin \{\mu\}$, φ violates either *2-unanimity* or *2-consistency*.

Let $N \in \mathcal{N}$, $e = (X, R) \in \mathcal{E}^N$, and $\mu \in Z(e)$. Let $\mathcal{H}(e, \mu) \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ be the set $\{i \succ_{xy} j \in \mathcal{X}^2 \times \mathcal{A}^2 : \{i, j\} \subset N, \{x, y\} \subset X, x\, P_i\, y, x\, P_j\, y, \mu_i = x, \mu_j = y\}$. In words, if φ is a *2-consistent* rule and $\varphi(e) = \mu$, then the set $\mathcal{H}(e, \mu)$ is contained in F^{φ}. For example, if $e = \left(R_i^{zxy}, R_j^{xyz}, R_k^{xzy}\right)$, $\mu_i = z$, $\mu_j = x$, and $\mu_k = y$, then $\mathcal{H}(e, \mu) = \{i \succ_{zy} k, j \succ_{xy} k\}$.

[16] The main theorem proved by Ehlers and Klaus [6] states only necessary conditions implied by *Pareto-efficiency*, *consistency*, and *strategy-proofness*.

[17] Observe that if $A(h, e) = \emptyset$, then if $h \subseteq F^{\varphi}$, φ violates either *2-unanimity* or *2-consistency*. Observe also that, if $|h| < \infty$, then $A(h, e)$ can be computed by inspection of each allocation $\mu \in Z(e)$ (see proof of Lemma 1 for one example.)

For each $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and each $e = (X, R) \in \mathcal{E}^N$, let $\mathcal{I}(h, e)$ be the set:[18]

$$\mathcal{I}(h,e) \equiv \begin{cases} \mathcal{X}^2 \times \mathcal{A}^2 & \text{if } |A(h,e)| = 0, \\ h \cup \mathcal{H}(e, A(h,e)) & \text{if } |A(h,e)| = 1, \\ h & \text{Otherwise.} \end{cases}$$

We call $\mathcal{I}(h, e)$ the set of implications of h by analyzing economy e. The following lemma justifies our choice for this name.

Lemma 4. *Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $e \in \mathcal{E}$. Let φ be a 2-unanimous and 2-consistent rule. If $h \subseteq F^\varphi$, then $\mathcal{I}(h, e) \subseteq F^\varphi$. If $h' \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ is such that $h \subseteq h'$, then $\mathcal{I}(h, e) \subseteq \mathcal{I}(h', e)$*

Proof. Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $e \in \mathcal{E}$. Observe that $\varphi(e) \in A(h, e)$ and thus $A(h, e) \neq \emptyset$. If $|A(h, e)| = 1$ and since φ is *2-unanimous* and *2-consistent*, then $\mathcal{H}(e, A(h, e)) \subseteq F^\varphi$. Thus, $\mathcal{I}(h, e) \subseteq F^\varphi$. Otherwise, $\mathcal{I}(h, e) = h \subseteq F^\varphi$. Now, let $h' \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ be such that $h \subseteq h'$. Observe that $A(h', e) \subseteq A(h, e)$. Hence, $\mathcal{I}(h, e) \subseteq \mathcal{I}(h', e)$. □

Observe that a consequence of Lemma 4 is that, if $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $e \in \mathcal{E}$ are such that $\mathcal{I}(h, e) = \mathcal{X}^2 \times \mathcal{A}^2$, then for each *2-unanimous* and *2-consistent* rule φ, $h \nsubseteq F^\varphi$.[19]

Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $\{e_k\}_{k=1}^K$ be a finite list of economies. We define now the set of implications of h by analyzing $\{e_k\}_{k=1}^K$. We denote it $\mathcal{I}(h, \{e_k\}_{k=1}^K)$. We do so by means of the following algorithm.

Algorithm 1. *Calculation of the implications of a set $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ by analyzing a finite list of economies $\{e_k\}_{k=1}^K$.*[20]
Input: $(h, \{e_k\}_{k=1}^K)$.
Output: $\mathcal{I}(h, \{e_k\}_{k=1}^K)$.
Step 1: *Let* $h_{K+1} \leftarrow h$.
Step 2: *Let* $t \leftarrow 1$ *and* $h_1 \leftarrow h_{K+1}$.
Step 3: *While* $t \leq K$, *let* $h_{t+1} \leftarrow \mathcal{I}(h_t, e_t)$ *and* $t \leftarrow t + 1$.
Step 4: *If* $h_{K+1} \setminus h_1 \neq \emptyset$, *then go to Step 2. Otherwise go to Step 5.*
Step 5: *Let* $\mathcal{I}(h, \{e_k\}_{k=1}^K) \leftarrow h_{K+1}$.

Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $\{e_k\}_{k=1}^K$ be a finite list of economies. Algorithm 1 with input $(h, \{e_k\}_{k=1}^K)$ finds the implications of h by analyzing the list of economies $\{e_k\}_{k=1}^K$. It proceeds as follows. First, it computes the implications of $h_1 = h$ by analyzing e_1. This

[18] We abuse of notation when $|A(h, e)| = 1$ and consider $A(h, e)$ as an allocation.
[19] Recall that F^φ is a family of *antisymmetric* binary relations.
[20] Let r, s be two variables; $r \leftarrow s$ indicates that r takes the value of s.

set is h_2. Then it computes the implications of h_2 by analyzing e_2, and so on. After finding the implications of h_K by analyzing e_K, h_{K+1}, the algorithm verifies if $h_{K+1} \setminus h_1 \neq \emptyset$. If this is the case, it returns to the previous step. Otherwise, the algorithm terminates and the set of implications of h by analyzing the list of economies $\{e_k\}_{k=1}^K$ is the last set calculated in the algorithm, i.e, h_{K+1} (see Figure 1 for a flowchart.)[21]

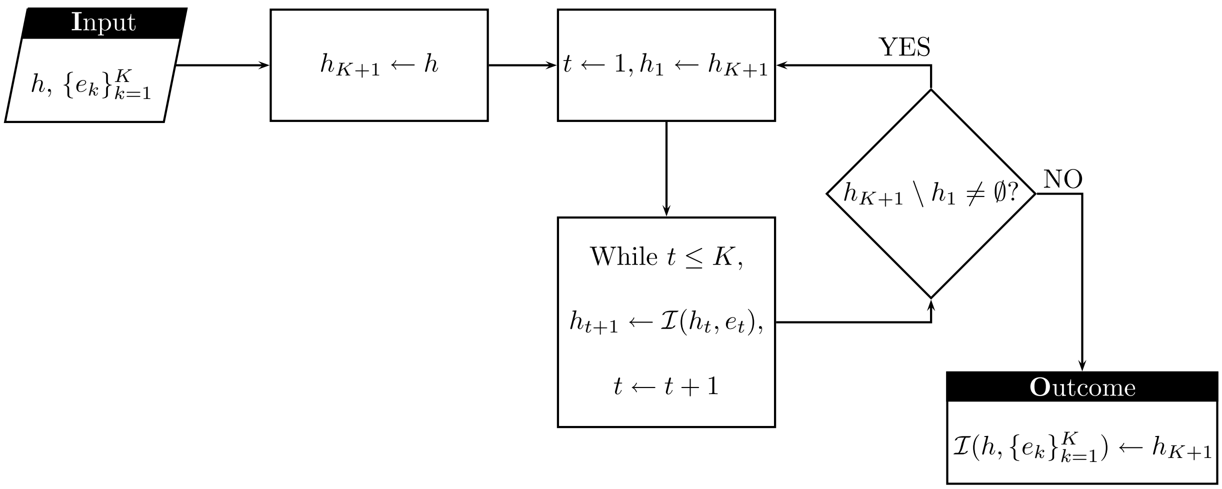

Figure 1: Flowchart for Algorithm 1.

The following remark guarantees that for each $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and each finite list of economies $\{e_k\}_{k=1}^K$, $\mathcal{I}(h, \{e_k\}_{k=1}^K)$ is well defined.

Remark 3. *Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $\{e_k\}_{k=1}^K$ be a finite list of economies. Then Algorithm 1 with h and $\{e_k\}_{k=1}^K$ as input, reaches Step 5.*

Proof. Observe that the algorithm terminates if and only if Step 4 is reached a finite number of times. Now, if the algorithm reaches Step 4 and $h_{K+1} = \mathcal{X}^2 \times \mathcal{A}^2$, then from Lemma 4, it can return to Step 4 at most once. Otherwise, Step 4 is reached at most $\sum_{k=1}^K |e_k|(|e_k| - 1)$ times.[22] □

Lemma 5. *Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $\{e_k\}_{k=1}^K$ be a finite list of economies. Let φ be a 2-unanimous and 2-consistent rule. If $h \subseteq F^\varphi$, then $\mathcal{I}\left(h, \{e_k\}_{k=1}^K\right) \subseteq F^\varphi$.*

[21] Observe that Algorithm 1 guarantees there is no $t \in \{1, \ldots, K\}$ such that $\mathcal{I}(\mathcal{I}(h, \{e_k\}_{k=1}^K), e_t) \setminus \mathcal{I}(h, \{e_k\}_{k=1}^K) \neq \emptyset$. That is, there are no "new" implications of $\mathcal{I}(h, \{e_k\}_{k=1}^K)$ that can be obtained by analyzing an economy listed in $\{e_k\}_{k=1}^K$.

[22] Observe that for each $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and each $e \in \mathcal{E}$ such that $|A(h, e)| = 1$, $|\mathcal{H}(e, A(h, e))| \leq |e|(|e| - 1)$.

Proof. Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and $\{e_k\}_{k=1}^K$ be a finite list of economies. Let φ be a *2-unanimous* and *2-consistent* rule. We claim that if $h \subseteq F^\varphi$, then $\mathcal{I}\left(h, \{e_k\}_{k=1}^K\right) \subseteq F^\varphi$. We use an inductive argument. Observe that at the first time that the algorithm (with h and $\{e_k\}_{k=1}^K$ as input) reaches Step 3, $t = 1$ and $h_t = h \subseteq F^\varphi$. Suppose now that at some step in the algorithm, $t \leq K$ and $h_t \subseteq F^\varphi$. By Lemma 4, $h_{t+1} = \mathcal{I}(h_t, e_t) \subseteq F^\varphi$. Hence, at the last time that the algorithm reaches Step 4, $h_{K+1} \subseteq F^\varphi$. Thus, $\mathcal{I}\left(h, \{e_k\}_{k=1}^K\right) \subseteq F^\varphi$. □

Observe that a consequence of Lemma 5 is that, if $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and the finite list of economies $\{e_k\}_{k=1}^K$ are such that $\mathcal{I}(h, \{e_k\}_{k=1}^K) = \mathcal{X}^2 \times \mathcal{A}^2$, then for each *2-unanimous* and *2-consistent* rule φ, $h \nsubseteq F^\varphi$.

The following lemma states that $\mathcal{I}$ is invariant under permutations of lists (second argument.)

Lemma 6. *Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$. If $\{e_k\}_{k=1}^K$ and $\{e'_k\}_{k=1}^K$ are two finite lists of economies such that $\bigcup_{k=1}^K e_k = \bigcup_{k=1}^K e'_k$, then $\mathcal{I}(h, \{e_k\}_{k=1}^K) = \mathcal{I}(h, \{e'_k\}_{k=1}^K)$.*

Proof. Let $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$. Let $\{e_k\}_{k=1}^K$ and $\{e'_k\}_{k=1}^K$ be two finite lists of economies such that $\bigcup_{k=1}^K e_k = \bigcup_{k=1}^K e'_k$. We claim that $\mathcal{I}(h, \{e_k\}_{k=1}^K) = \mathcal{I}(h, \{e'_k\}_{k=1}^K)$. We prove first that $\mathcal{I}(h, \{e_k\}_{k=1}^K) \subseteq \mathcal{I}(h, \{e'_k\}_{k=1}^K)$. We use an inductive argument. Observe that at the first time that the algorithm (with h and $\{e_k\}_{k=1}^K$ as input) reaches Step 2, $h_1 = h \subseteq \mathcal{I}(h, \{e'_k\}_{k=1}^K)$. Now, suppose that at some step in the algorithm (with h and $\{e_k\}_{k=1}^K$ as input), $t \leq K$ and $h_t \subseteq \mathcal{I}(h, \{e'_k\}_{k=1}^K)$. As the algorithm (with h and $\{e'_k\}_{k=1}^K$ as input) reaches Step 4, then it has to be the case that $\mathcal{I}(\mathcal{I}(h, \{e'_k\}_{k=1}^K), e_t) = \mathcal{I}(h, \{e'_k\}_{k=1}^K)$. From Lemma 4, $h_{t+1} \subseteq \mathcal{I}(h, \{e'_k\}_{k=1}^K)$.[23] Thus, $\mathcal{I}(h, \{e_k\}_{k=1}^K) \subseteq \mathcal{I}(h, \{e'_k\}_{k=1}^K)$. A symmetric argument shows that $\mathcal{I}(h, \{e'_k\}_{k=1}^K) \subseteq \mathcal{I}(h, \{e_k\}_{k=1}^K)$. □

For each $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and each $E \subseteq \mathcal{E}$ such that $|E| < \infty$, the set of implications of h by analyzing E is $\mathcal{I}(h, E) \equiv \mathcal{I}\left(h, \{e_k\}_{k=1}^K\right)$ for some list $\{e_k\}_{k=1}^K$ such that $E = \bigcup_{k=1}^K e_k$. From Lemma 6, $\mathcal{I}(h, E)$ is well defined. From Lemma 5, for each $h \subseteq \mathcal{X}^2 \times \mathcal{A}^2$ and each $E \subseteq \mathcal{E}$ such that $|E| < \infty$, if $h \subseteq F^\varphi$, then $\mathcal{I}(h, E) \subseteq F^\varphi$. Besides, if $\mathcal{I}(h, E) = \mathcal{X}^2 \times \mathcal{A}^2$, then $h \nsubseteq F^\varphi$.

For each $N \in \mathcal{N}$ and $E \subseteq \mathcal{E}$ such that $|E| < \infty$, let E_N^X be the set of economies $E_N^X \equiv \{e = (X', R) \in \mathcal{E}^{N'} : N' \subseteq N,\ X' \subseteq X, \text{ and } |N'| = |X'| = 3\}$.

[23] Here h_{t+1} refers to the set calculated the next time the algorithm (with h and $\{e_k\}_{k=1}^K$ as input), reaches Step 3.

Example 7. *If* $h_0 \equiv \{i \succ_{xy} j, j \succ_{xy} k, j \succ_{xz} i\}$*, then* $\mathcal{I}(h_0, E^{\{x,y,z,w\}}_{\{i,j,k\}}) = \mathcal{X}^2 \times \mathcal{A}^2$.[24]

7 Appendix B

Proof of Lemma 2. Assume that $|\mathcal{X}| \geq 4$, $i \succ^{\varphi}_{xy} j$, and $j \succ^{\varphi}_{xy} k$. We claim that, $i \succ^{\varphi}_{xy} k$. Suppose by contradiction that $k \succ^{\varphi}_{xy} i$. Let $\{z, w\} \subseteq \mathcal{X} \setminus \{x, y\}$ be such that $z \neq w$. From Lemma 1, $h \equiv \{i \succ^{\varphi}_{zy} j,\ i \succ^{\varphi}_{zy} k,\ j \succ^{\varphi}_{wy} k\} \subseteq F^{\varphi}$. Observe that $\mathcal{I}(h, (R^{zyw}_i, R^{wzy}_j, R^{zwy}_k)) = \mathcal{X}^2 \times \mathcal{A}^2 \subseteq F^{\varphi}$. This is a contradiction. Now, we claim that $i \succ^{\varphi}_{xz} k$. To prove this, let $h' \equiv \{i \succ_{xy} j,\ j \succ_{xy} k,\ i \succ_{xy} k\}$ and observe that $\{i \succ_{xz} k\} = \mathcal{I}(h', (R^{xyz}_i, R^{xyz}_j, R^{xzy}_k))$. It is left to prove that $i \succ^{\varphi}_{xz} j$. To do this, suppose by contradiction that $i \succ^{\varphi}_{xz} j$. Let $h_0 \equiv \{i \succ_{xy} j,\ j \succ_{xy} k,\ j \succ_{xz} i\}$. From Example 7, $\mathcal{I}(h_0, E^{\{x,y,z,w\}}_{\{i,j,k\}}) = \mathcal{X}^2 \times \mathcal{A}^2 \subseteq F^{\varphi}$. This is a contradiction. □

Proof of Lemma 3. Assume $|\mathcal{A}| \geq 4$. Let φ be a *2-unanimous* and *2-consistent* rule, and $\{i, j, k, l\} \subseteq \mathcal{A}$. Suppose that for some $\{x, y\} \subseteq \mathcal{X}$ such that $x \neq y$, $i \succ^{\varphi}_{xy} j \succ^{\varphi}_{xy} k \succ^{\varphi}_{xy} l$. We want to prove $i \succ^{\varphi} \{k, l\}$. From Lemma 2, $i \succ^{\varphi}_{x} j \succ^{\varphi}_{x} \{k, l\}$. Let $\{z, v\} \subseteq \mathcal{X} \setminus \{x\}$ be such that $z \neq v$. Hence, $i \succ^{\varphi}_{xz} j \succ^{\varphi}_{xz} \{k, l\}$, and from Lemma 1, $i \succ^{\varphi}_{vz} \{k, l\}$. It remains to prove that for each $z \in X \setminus \{x\}$, $i \succ^{\varphi}_{zx} \{k, l\}$. Let $z \in X \setminus \{x, y\}$ and $h_1 \equiv \left\{i \succ^{\varphi}_{zy} k, k \succ^{\varphi}_{xy} l\right\}$. Observe that $\mathcal{I}\left(h_1, (R^{zxy}_i, R^{zxy}_k, R^{xyz}_l)\right) = \{i \succ^{\varphi}_{zx} k\}$. Now, let $h_2 \equiv \left\{i \succ^{\varphi}_{zx} k, i \succ^{\varphi}_{yz} l\right\}$. Observe that $\mathcal{I}\left(h_2, (R^{yzx}_i, R^{zxy}_k, R^{yxz}_l)\right) = \left\{i \succ^{\varphi}_{yx} l\right\}$.

We now claim that for each $z \in X \setminus \{x, y\}$, $i \succ^{\varphi}_{zx} l$. To prove this, suppose by contradiction that $l \succ^{\varphi}_{zx} i$. Let $h_3 \equiv \left\{i \succ^{\varphi}_{zx} k, i \succ^{\varphi}_{zy} k, l \succ^{\varphi}_{zx} i\right\}$. Observe that $\mathcal{I}\left(h_3, (R^{zxy}_i, R^{zyx}_k, R^{yzx}_l)\right) = \left\{l \succ^{\varphi}_{yx} k\right\}$. Now, let $h_4 \equiv \left\{i \succ^{\varphi}_{zy} l, l \succ^{\varphi}_{zx} i\right\}$. Observe that $\mathcal{I}\left(h_4, (R^{zyx}_i, R^{yxz}_k, R^{zxy}_l)\right) = \left\{k \succ^{\varphi}_{yx} i\right\}$. Thus, $k \succ^{\varphi}_{yx} i \succ^{\varphi}_{yx} l \succ^{\varphi}_{yx} k$. This contradicts Lemma 2.

Finally, we claim that $i \succ^{\varphi}_{yx} k$. To prove this, suppose by contradiction that $k \succ^{\varphi}_{yx} i$. Let $h_5 \equiv \left\{ \begin{array}{c} k \succ^{\varphi}_{yx} i, i \succ^{\varphi}_{zx} l, \\ i \succ^{\varphi}_{yz} l, i \succ^{\varphi}_{yz} k \end{array} \right\}$. Observe that $\mathcal{I}\left(h_5, (R^{yzx}_i, R^{yxz}_k, R^{yzx}_l)\right) = \mathcal{X}^2 \times \mathcal{A}^2$. This is a contradiction. □

Proof of Theorem 1 for $|\mathcal{A}| \leq 3$***.*** Assume $|\mathcal{A}| \leq 3$ and $|\mathcal{X}| \geq 4$. We prove that if a rule φ is *3-unanimous* and *3-consistent*, then it is an MDPEwt rule. That is, there is an MDPE structure with tail, $(\Pi, \succ, T, Q)$, such that $\varphi = D^{(\Pi, \succ, T, Q)}$.

[24] Appendix B contains a "step by step" computation of $\mathcal{I}(h_0, E^{\{x,y,z,w\}}_{\{i,j,k\}})$. Software implementing Algorithm 1 is available at http://troi.cc.rochester.edu/˜vlez/consistency and upon request to the corresponding author.

Suppose $|\mathcal{A}| \leq 2$. There are two possible cases: φ is a tail rule or φ is a serial dictatorship. In both cases clearly φ is an MDPEwt rule.

Suppose $|\mathcal{A}| = 3$, say $\mathcal{A} = \{i, j, k\}$. For each $n \in \mathcal{A}$, let $O(n) \equiv \{x \in \mathcal{X} : n \succ_x^\varphi \mathcal{A} \setminus \{n\}\}$. Observe that for each $\{n, n'\} \subseteq \mathcal{A}$ such that $n \neq n'$, $O(n) \bigcap O(n') = \emptyset$. We claim that $\bigcup_{n \in \mathcal{A}} O(n) = \mathcal{X}$. To prove this, let $x \in \mathcal{X}$ and $z \in \mathcal{X} \setminus \{x\}$. As $|\mathcal{X}| \geq 4$, then $\succ_{xz}^\varphi$ is transitive (see note after Lemma 2). Thus, there is $n_x \in \mathcal{A}$ such that $n_x \succ_{xz}^\varphi \mathcal{A} \setminus \{n_x\}$. From Lemma 2, $n_x \succ_x^\varphi \mathcal{A} \setminus \{n_x\}$ and $x \in \bigcup_{n \in \mathcal{A}} O(n)$.

Now, we claim that $|\{n \in \mathcal{A} : O(n) \neq \emptyset\}| \leq 2$. To prove this, suppose by contradiction that there are $\{x, y, z\} \subseteq \mathcal{X}$ such that $x \in O(i)$, $y \in O(j)$, and $z \in O(k)$. Let $w \in \mathcal{X} \setminus \{x, y, z\}$ and $h \equiv \{i \succ_{xw} j, i \succ_{xw} k, j \succ_{yw} i\}$. Observe that $\{j \succ_{xw} k\} \subseteq \mathcal{I}\left(h, \left(R_i^{yxw}, R_j^{xyw}, R_k^{xyw}\right)\right)$. Thus, from Lemma 1, $i \succ_x^\varphi j \succ_{xw}^\varphi k$ implies $i \succ_{zw}^\varphi k$. This contradicts $z \in O(k)$.

There are three cases.

Case 1 $|\{n \in \mathcal{A} : O(n) \neq \emptyset\}| = 1$. Suppose w.l.o.g. that $\{i\} \succ^\varphi \{j, k\}$. If there is $n' \in \{j, k\}$ such that $n' \succ^\varphi \{j, k\} \setminus \{n'\}$, then φ is a serial dictatorship and thus, an MDPEwt rule. Otherwise, let $(\Pi, \succ) \equiv \{i\} \succ \{j, k\}$ and $Q = \varphi | \bigcup_{N \subseteq \{j,k\}} \mathcal{E}^N$. Observe that since φ is *2-unanimous* and *2-consistent*, then $\varphi = D^{\left(\Pi, \succ, T^\emptyset, Q\right)}$.

Case 2 $|\{n \in \mathcal{A} : O(n) \neq \emptyset\}| = 2$ and there is $n' \in \mathcal{A}$ such that $\mathcal{A} \setminus \{n'\} \succ^\varphi \{n'\}$. Suppose w.l.o.g. that $\{i, j\} \succ^\varphi \{k\}$. Let $(\Pi, \succ) \equiv \{i, j\} \succ \{k\}$, and for each $x \in \mathcal{X}$, let $T(\{i, j\}, x) = i$ if and only if $x \in O(i)$. As φ is *2-unanimous* and *2-consistent*, then $\varphi = D^{\left(\Pi, \succ, T, Q^\emptyset\right)}$.

Case 3 $|\{n \in \mathcal{A} : O(n) \neq \emptyset\}| = 2$, and there is no $n' \in \mathcal{A}$ such that $\mathcal{A} \setminus \{n'\} \succ^\varphi \{n'\}$. Suppose w.l.o.g that $\{n \in \mathcal{A} : O(n) \neq \emptyset\} = \{i, j\}$. Since it is not true that $\{i, j\} \succ^\varphi \{k\}$, then there are $n \in \{i, j\}$ and $\{x, y^*\} \subseteq \mathcal{X}$ with $x \neq y^*$, such that $k \succ_{xy^*}^\varphi n$. Suppose w.l.o.g. that $n = j$. Let $\gamma : \{1, 2, 3\} \to \mathcal{A}$ be such that $\gamma(1) = i$, $\gamma(2) = j$, and $\gamma(3) = k$. Let $Y \equiv \left\{z \in \mathcal{X} : k \succ_{zy^*}^\varphi j\right\}$. Observe that $Y \neq \emptyset$. Let $E \equiv \{e \in \mathcal{E}_{\gamma, y^*, Y} : \varphi_k(e) P_k \{\varphi_i(e), \varphi_j(e)\}\}$.

We claim that $O(i) = \mathcal{X} \setminus \{y^*\}$ and $O(j) = \{y^*\}$. To prove this, let $x \in \mathcal{X}$ be such that $k \succ_{xy^*} j$ and $z \in \mathcal{X} \setminus \{x, y^*\}$. Hence, $x \in O(i)$. Now, since $i \succ_{xy^*} k \succ_{xy^*} j$, then from Lemma 1, $i \succ_{xy^*} j$ and $z \in O(i)$. Thus, $O(i) \supseteq \mathcal{X} \setminus \{y^*\}$ and since $O(j) \neq \emptyset$, then $O(j) = \{y^*\}$.

We claim that $\varphi = \mathcal{T}^{(\gamma, y^*, Y, E)}$. We prove first that for each $N \in \mathcal{N}$ such that $|N| = 2$, and each $e \in \mathcal{E}^N$, $\varphi(e) = \mathcal{T}^{(\gamma, y^*, Y, E)}(e)$. By *2-unanimity*, we just have to verify this equality for the economies where both agents have the same preference. Moreover, since $O(i)=\mathcal{X}\setminus\{y^*\}$, $O(j)=\{y^*\}$, and by construction of Y, we just have to prove that for each $z\in\mathcal{X}\setminus\{y^*\}$, $i\succ^{\varphi}_{y^*z}k$, and for each $\{v,z\}\subseteq\mathcal{X}\setminus\{y^*\}$ such that $v\neq z$, $j\succ^{\varphi}_{vz}k$. Let $h\equiv\{i\succ_{vz}j, i\succ_{vz}k, j\succ_{y^*z}k\}$. Observe that, $\mathcal{I}\left(h,\left(R_i^{yvz},R_j^{vyz},R_k^{vyz}\right)\right)=\{i\succ_{y^*z}k, j\succ_{vz}k\}$.

Now, we prove that for each three-agent economy $e=(X,R)\in\mathcal{E}^{\mathcal{A}}$, $\varphi(e)=\mathcal{T}^{(\gamma,y^*,Y,E)}(e)$. There are nine subcases. **Subcase 3.1**: $y^*\notin X$. In this subcase, since $O(i)=X\setminus\{y^*\}$, then by *2-unanimity* and *2-consistency*, $\varphi_i(e)=D_i^{i\succ j\succ k}(e)$. Now, by *2-consistency*$\varphi(e)|_{\{j,k\}}=\varphi(r^{\varphi(e)}_{\{j,k\}}(e))=D^{i\succ j\succ k}(e)|_{\{j,k\}}$. **Subcase 3.2**: $X=\{x,y^*,z\}$, $x\,P_i\,X\setminus\{x\}$, $z\,P_k\,y^*$, and $z\in Y$. Using a similar argument as in Subcase 3.1, $\varphi_i(e)=D_i^{i\succ k\succ j}(e)$ and $\varphi(e)|_{\{j,k\}}=\varphi(r^{\varphi(e)}_{\{j,k\}}(e))=D^{i\succ k\succ j}(e)|_{\{j,k\}}$. **Subcase 3.3**: $X=\{x,y^*,z\}$, $x\,P_i\,X\setminus\{x\}$, and one of the two following statements holds: $z\,P_k\,y^*$ and $z\notin Y$, or $y^*\,P_k\,z$. Using a similar argument as in Subcase 3.1, $\varphi_i(e)=D_i^{i\succ j\succ k}(e)$ and $\varphi(e)|_{\{j,k\}}=\varphi(r^{\varphi(e)}_{\{j,k\}}(e))=D^{i\succ j\succ k}(e)|_{\{j,k\}}$. **Subcase 3.4**: $X=\{x,y^*,z\}$ and $y^*\,P_{\{i,j\}}\,X\setminus\{y^*\}$. Using a similar argument as in Subcase 3.1, $\varphi_j(e)=D_j^{j\succ i\succ k}(e)$ and $\varphi(e)|_{\{i,k\}}=\varphi(r^{\varphi(e)}_{\{i,k\}}(e))=D^{j\succ i\succ k}(e)|_{\{i,k\}}$. **Subcase 3.5**: $X=\{x,y^*,z\}$, $y^*\,P_i\,X\setminus\ \{y^*\}$, and $x\,P_j\,y^*$. Observe that by *2-unanimity* and *2-consistency*, and since $i\succ^{\varphi}k$, then $\varphi_i(e)\in\{y^*,x\}$ and $\varphi_k(e)\neq y^*$. We claim that $\varphi_i(e)=D_i^{i\succ j\succ k}(e)=y^*$. To prove this, suppose by contradiction that $\varphi_i(e)=x$. Hence, $\varphi_k(e)=z$ and $\varphi_j(e)=y^*$. Thus, by *2-consistency*$\varphi(r^{\varphi(e)}_{\{i,j\}}(e))=\varphi(e)|_{\{i,j\}}\notin U(r^{\varphi(e)}_{\{i,j\}}(e))\neq\emptyset$. This is a contradiction. Now, by a similar argument as in Subcase 3.1, $\varphi(e)|_{\{j,k\}}=\varphi(r^{\varphi(e)}_{\{j,k\}}(e))=D^{i\succ j\succ k}(e)|_{\{j,k\}}$. **Subcase 3.6**: $X=\{x,y^*,z\}$, $y^*\,P_i\,X\setminus\{y^*\}$, $z\,P_j\,y^*\,P_j\,x$, and $y^*\,P_k\,z$. Using a similar argument as in Subcase 3.5, $\varphi_i(e)\in\{y^*,x\}$ and $\varphi_k(e)\neq y^*$. We claim that $\varphi_i(e)=D_i^{i\succ j\succ k}(e)=y^*$. To prove this, suppose by contradiction that $\varphi_i(e)=x$. Hence, $\varphi_k(e)=z$ and $\varphi_j(e)=y^*$. Thus, by *2-consistency*$\varphi(r^{\varphi(e)}_{\{j,k\}}(e))=\varphi(e)|_{\{j,k\}}\notin U(r^{\varphi(e)}_{\{j,k\}}(e))\neq\emptyset$. This is a contradiction. Now, by a similar argument as in Subcase 3.1, $\varphi(e)|_{\{j,k\}}=\varphi(r^{\varphi(e)}_{\{j,k\}}(e))=D^{i\succ j\succ k}(e)|_{\{j,k\}}$. **Subcase 3.7**: $X=\{x,y^*,z\}$, $y^*\,P_i\,X\setminus\{y^*\}$, $z\,P_j\,y^*\,P_j\,x$, and $x\,P_k\,z\,P_k\,y^*$. Using a similar argument as in Subcase 3.5, $\varphi_i(e)\in\{y^*,x\}$ and $\varphi_k(e)\neq y^*$. We claim that $\varphi_i(e)=D_i^{i\succ j\succ k}(e)=y^*$. To prove this, suppose by contradiction that $\varphi_i(e)=x$. Hence, $\varphi_k(e)=z$ and $\varphi_j(e)=y^*$, and $\varphi(e)\notin U(e)\neq\emptyset$. This contradicts *3-unanimity*.[25] Now, by a similar argument as in Subcase 3.1, $\varphi(e)|_{\{j,k\}}=\varphi(r^{\varphi(e)}_{\{j,k\}}(e))=D^{i\succ j\succ k}(e)|_{\{j,k\}}$. **Subcase 3.8**: $X=\{x,y^*,z\}$, $y^*\,P_i\,X\setminus\{y^*\}$, $z\,P_j\,y^*\,P_j\,x$, $z\,P_k\,X\setminus\{z\}$, and one of

[25]This is the only step in the proof where it is required to use *3-unanimity* and it is not enough *2-unanimity*.

the following holds: $z \notin Y$ or $e \notin E$. Using a similar argument as in Subcase 3.5, $\varphi_k(e) \neq y^*$. We claim that $\varphi_k(e) = D_k^{i \succ j \succ k}(e) = x$. Suppose by contradiction that $z \notin Y$ and $\varphi_k(e) = z$. Thus, as $\varphi_j(e) \in \{x, y^*\}$, $\varphi(e)|_{\{j,k\}} \neq \varphi(r_{\{j,k\}}^{\varphi(e)}(e))$. This contradicts *2-consistency*. Now, by a similar argument as in Subcase 3.1, $\varphi(e)|_{\{i,j\}} = \varphi(r_{\{i,j\}}^{\varphi(e)}(e)) = D^{i \succ j \succ k}(e)|_{\{i,j\}}$. **Subcase 3.9**: $X = \{x, y^*, z\}$, $y^* \, P_i \, X \setminus \{y^*\}$, $z \, P_j \, y^* \, P_j \, x$, $z \, P_k \, X \setminus \{z\}$, $z \in Y$, and $e \in E$. Observe that as $e \in E$, then $\varphi_k(e) = z$. In this subcase, $\varphi_k(e) = D^{k \succ j \succ i}(e)(k) = z$. Using a similar argument as in Subcase 3.1, $\varphi(e)|_{\{i,j\}} = \varphi(r_{\{i,j\}}^{\varphi(e)}(e)) = D^{k \succ j \succ i}(e)|_{\{i,j\}}$.

□

Proof of Theorem 1 for $|\mathcal{A}| \geq 4$. We prove Theorem 1 in the case $|\mathcal{N}| < \infty$. The proof extends easily to the infinite case. We want to prove that if a rule φ is *3-unanimous* and *3-consistent*, then it is an MDPEwt rule. That is, there is an MDPEwt structure $(\Pi, \succ, T, Q)$ such that $\varphi = D^{(\Pi, \succ, T, Q)}$. Suppose that $|\mathcal{A}| \geq 4$. We first identify a candidate MDPE structure with tail, $(\Pi, \succ, T, Q)$, and then we prove $\varphi = D^{(\Pi, \succ, T, Q)}$.

Let $\{i, j\} \subseteq \mathcal{A}$. Fix a pair $\{x, y\} \subseteq \mathcal{X}$ such that $x \neq y$. Suppose w.l.o.g. that $i \succ_{xy}^{\varphi} j \succ_{xy}^{\varphi} \mathcal{A}/\{i, j\}$. Lemma 3 implies $i \succ^{\varphi} \mathcal{A}/\{i, j\}$. There are two possible cases.

Case 1 $i \succ^{\varphi} j$. In this case, let $\pi_1 \equiv \{i\}$.

Case 2 There is a pair $\{v, w\} \subseteq \mathcal{X}$ such that $v \neq w$, and $j \succ_{vw}^{\varphi} i \succ^{\varphi} \mathcal{A}/\{i, j\}$. Lemma 3 implies that $j \succ^{\varphi} \mathcal{A}/\{i, j\}$. In this case, let $\pi_1 \equiv \{i, j\}$, and for each $z \in \mathcal{X}$, let $T(\pi_1, z) \equiv i$ if and only if $i \succ_z^{\varphi} j$. As $\mathcal{A} \setminus \pi_1 \neq \emptyset$, Lemma 2 implies T is well defined.

Suppose now that $\pi_1, \pi_2, ..., \pi_{t-1}$, have been constructed, and for each $\pi \in \{\pi_k\}_{k=1}^{t-1}$ such that $|\pi| = 2$, and each $z \in \mathcal{X}$, $T(\pi, z)$ has been defined. Let $\mathcal{A}' \equiv \mathcal{A}/\bigcup_{s=1}^{t-1} \pi_s$. Assume $\mathcal{A}' \neq \emptyset$. There are two possible cases.

Case 1 $|\mathcal{A}'| \geq 4$. Suppose w.l.o.g. $i', j' \in \mathcal{A}'$ are such that $i' \succ_{xy}^{\varphi} j' \succ_{xy}^{\varphi} \mathcal{A}'/\{i', j'\}$. Following the same argument as in Case 2 above, if $i' \succ^{\varphi} \mathcal{A}'/\{i'\}$, let $\pi_t \equiv \{i'\}$. Otherwise, let $\pi_t \equiv \{i', j'\}$, and for each $z \in \mathcal{X}$, let $T(\pi_t, z) \equiv i'$ if and only if $i' \succ_z^{\varphi} j'$.

Case 2 $|\mathcal{A}'| \leq 3$. Let $\varphi|_{\mathcal{A}'}$ be the restriction of φ to the set of economies $\bigcup_{N \subseteq \mathcal{A}'} \mathcal{E}^N$. That is, for each $e \in \bigcup_{N \subseteq \mathcal{A}'} \mathcal{E}^N$, $\varphi|_{\mathcal{A}'}(e) = \varphi(e)$. Since φ is *2-unanimous* and *consistent* so is $\varphi|_{\mathcal{A}'}$. Since $\varphi|_{\mathcal{A}'}$ is defined for a set of potential agents with cardinality, at most, three, then there is a MDPE structure with tail, $(\Pi', \succ', T', Q')$, such that $\varphi|_{\mathcal{A}'} = D^{(\Pi', \succ', T', Q')}$ (see the proof of Theorem 1 for $|\mathcal{A}| \leq 3$.) Suppose w.l.o.g that

$(\Pi', \succ') = \pi'_1 \succ ... \succ \pi'_K$. Observe that $1 \leq K \leq 3$ and $\tau(\Pi', \succ') = \pi'_K$. For each $k = 1, .., K$, let $\pi_{t-1+k} \equiv \pi'_k$. If $|\mathcal{A}'| = 3$ and $|\{\pi'_1\}| = 2$, then for each $z \in \mathcal{X}$, let $T(\pi_t, z) = T'(\pi'_1, z)$.

Since $|\mathcal{A}| < \infty$, the process described above defines a partition $\Pi = \{\pi_k\}_{k=1,...,I}$ of $\mathcal{A}$ and a tie-breaker T on Π_2. Let $\succ$ be the linear order on Π given by $\pi_1 \succ \pi_2 \succ ... \succ \pi_I$. By construction, if $i \in \pi_k$, $j \in \pi_{k'}$, and $\pi_k \succ \pi_{k'}$, then $i \succ^{\varphi} j$. Clearly, T is a tie-breaker defined on Π_2. Let Q be the restriction of φ to the set of economies $\bigcup_{N \subseteq \pi_I} \mathcal{E}^N$. Observe that $(\Pi, \succ, T, Q)$ is an MDPE structure with tail.

We claim that $\varphi = D^{(\Pi,\succ,T,Q)}$. To see this, let $N \in \mathcal{N}$ and $e = (X, R) \in \mathcal{E}^N$. We want to show $\varphi(e) = D^{(\Pi,\succ,T,Q)}(e)$. Let $(\Pi^N, \succ^N)$ be the ordered partition induced by $(\Pi, \succ)$ on N. Suppose w.l.o.g that π_1^N is the maximal set in Π^N w.r.t. $\succ^N$, and that $\pi_1^N \cap \tau(\Pi, \succ) = \emptyset$. There are two cases.

Case 1 $|\pi_1^N| = 1$. Suppose w.l.o.g. that $\pi_1^N = \{i\}$. We claim that $\varphi_i(e) = D_i^{(\Pi,\succ,T,Q)}(e)$. Suppose by contradiction that there is $j \in N \setminus \{i\}$ such that $\varphi_j(e) \, P_i \, \varphi_i(e)$. By *2-consistency*, $j \succ^{\varphi}_{\varphi(e)(j)\varphi(e)(i)} i$. This is a contradiction.

Case 2 $|\pi_1^N| = 2$. Suppose $\pi_1^N = \{i, j\}$. Using a similar argument as in Case 1, for each $k \in N \setminus \{i, j\}$, $\varphi_i(e) \, P_i \, \varphi_k$ and $\varphi_j(e) \, P_j \, \varphi_k(e)$. By *2-consistency*, $\varphi(e)|_{\pi_1^N} = \varphi(r^{\varphi(e)}_{\{i,j\}}(e))$. Moreover, by construction $\varphi(r^{\varphi(e)}_{\{i,j\}}(e)) = D^{(\Pi,\succ,T,Q)}(r^{\varphi(e)}_{\{i,j\}}(e))$. Thus, $\varphi(e)|_{\pi_1^N} = D^{(\Pi,\succ,T,Q)}(e)|_{\pi_1^N}$.

Suppose now that $\varphi(e)|_{\bigcup_{k=1}^s \pi_k^N} = D^{(\Pi,\succ,T,Q)}|_{\bigcup_{k=1}^s \pi_k^N}$, where $\bigcup_{k=1}^s \pi_k^N \succ N \setminus \bigcup_{k=1}^s \pi_k^N$. Let π_{s+1}^N be the maximal set on $\Pi^N \setminus \bigcup_{k=1}^s \pi_k^N$ w.r.t. $\succ^N$. If $\pi_{s+1}^N \cap \tau(\Pi, \succ) = \emptyset$, a similar argument shows that $\varphi(e)|_{\pi_{s+1}^N} = D^{(\Pi,\succ,T,Q)}(e)|_{\pi_{s+1}^N}$. If $\pi_{s+1}^N \subseteq \tau(\Pi, \succ)$, then by *3-consistency*, $\varphi(e)|_{\pi_{s+1}^N} = \varphi(r^{\varphi(e)}_{\pi_{s+1}^N}(e))$.[26] Thus, $\varphi(e)|_{\pi_{s+1}^N} = Q(e)|_{\pi_{s+1}^N} = D^{(\Pi,\succ,T,Q)}|_{\pi_{s+1}^N}$. □

Proof of Remark 1. If $|\mathcal{A}| \leq 2$, the proof is trivial. Suppose that $\mathcal{A} = \{i, j, k\}$. Let (γ, y^*, Y, E) be a tail structure. We want to prove that $\mathcal{T}^{(\gamma,y^*,Y,E)}$ is *Pareto-efficient* and *consistent*. We use the shorthand $\mathcal{T}$ for $\mathcal{T}^{(\gamma,y^*,Y,E)}$ hereafter. Let $e \in \mathcal{E}$. Observe that $\mathcal{T}(e) = D^{\succ_e}$ for some linear order $\succ_e$. Thus, $\mathcal{T}(e) \in P(e)$. Now, we prove $\mathcal{T}$ is *consistent*. We have to prove that for each $e \in \mathcal{E}^{\mathcal{A}}$, and each $N \in \mathcal{N}$, $|N| = 2$, $\mathcal{T}(e)|_N = \mathcal{T}(r_N^{\mathcal{T}(e)}(e))$. Observe first that this is true in case e falls in the exceptions listed in Table 1 (this can be seen by inspection of the allocations in each case.) Otherwise, $\mathcal{T}(e) = D^{\gamma(1)\succ\gamma(2)\succ\gamma(3)}(e)$. Let $\succ^*$ be the linear order given by $\gamma(1) \succ^* \gamma(2) \succ^* \gamma(3)$.

[26] This is the only step in the proof where it is required *3-consistency* of φ, because $|\tau(\Pi, \succ)|$ could be potentially equal to 3.

Observe that, by definition of $\mathcal{T}$, $\mathcal{T}(r^{\mathcal{T}(e)}_{\{\gamma(1),\gamma(3)\}}(e)) = D^{\succ^*}(r^{\mathcal{T}(e)}_{\{\gamma(1),\gamma(3)\}}(e))$. Now, we claim that $\mathcal{T}(r^{\mathcal{T}(e)}_{\{\gamma(1),\gamma(2)\}}(e)) = D^{\succ^*}(r^{\mathcal{T}(e)}_{\{\gamma(1),\gamma(2)\}}(e))$. Suppose by contradiction that this is not true. Thus, $X = \{x, y^*, z\}$, $r^{\mathcal{T}(e)}_{\{\gamma(1),\gamma(2)\}}(e) = (\{y^*, z\}, (y^* P_{\gamma(1)} z, y^* P_{\gamma(1)} z))$, and $\mathcal{T}_{\gamma(1)}(r^{\mathcal{T}(e)}_{\{\gamma(1),\gamma(2)\}}(e)) = z$. By definition of $\mathcal{T}$, $\gamma(2)$ never gets an object that is worse than y^*. Thus, $\mathcal{T}_{\gamma(2)}(e) = y^*$. This is a contradiction, because $\mathcal{T}(e) = D^{\succ^*}(e)$. A similar argument shows that $\mathcal{T}(r^{\mathcal{T}(e)}_{\{\gamma(2),\gamma(3)\}}(e)) = D^{\succ^*}(r^{\mathcal{T}(e)}_{\{\gamma(2),\gamma(3)\}}(e))$. □

Proof of Remark 2. Let $(\Pi, \succ, T, Q)$ be an MDPE structure with tail. We want to prove that $D^{(\Pi,\succ,T,Q)}$ is *Pareto-efficient* and *consistent*. We use the shorthand D for $D^{(\Pi,\succ,T,Q)}$ hereafter. Let $N \in \mathcal{N}$ and $e = (X, R) \in \mathcal{E}^N$. Observe that, there is a linear order on $\mathcal{N}$, $\succ_e$, such that $D(e) = D^{\succ_e}(e)$. As serial dictatorships are *Pareto-efficient*, then $D(e) \in P(e)$. Let $N' \subseteq N$. We claim that $D(r^{D(e)}_{N'}(e)) = D(e)|_{N'}$. To prove this, suppose by contradiction that $D(r^{D(e)}_{N'}(e)) \neq D(e)|_{N'}$. Let $\pi^{N'} \in \Pi^{N'}$ be such that for each $k \succ \pi^{N'}$, $D_k(r^{D(e)}_{N'}(e)) = D_k(e)$, and for some $i \in \pi^{N'}$, $D_i(r^{D(e)}_{N'}(e)) \neq D_i(e)$. Observe that

$$\bigcup_{k\in\pi^{N'}} D_k(e) \subseteq \bigcup_{k\in N'} D_k(e) \setminus \bigcup_{k\succ\pi^{N'},k\in N'} D_k(e) = \bigcup_{k\in N'} D_k(r^{D(e)}_{N'}(e)) \setminus \bigcup_{k\succ\pi^{N'},k\in N'} D_k(r^{D(e)}_{N'}(e)).$$

Thus, $D(e)|_{\pi^{N'}} = D(r^{D(e)}_{N'}(e))|_{\pi^{N'}}$. This is a contradiction. □

Proof of Proposition 1. Let $\mathcal{A} = \{i, j, k\}$. Consider a tail rule $\mathcal{T}^{(\gamma,y,Y,E)}$. We want to prove that $\mathcal{T}^{(\gamma,y,Y,E)}$ is *strategy-proof* if and only if $E = \emptyset$. We prove first that if $\mathcal{T}^{(\gamma,y,Y,E)}$ is *strategy-proof*, then $E = \emptyset$. Suppose by contradiction that $E \neq \emptyset$. Suppose w.l.o.g. that $\gamma(2) = j$. Let $e = (X, R) \in E$, $X = \{x, y^*, z\}$, and $z \in Y$. Recall that $x\, P_j\, y^*\, P_j\, z$. Let R'_j be such that $x\, P'_j\, z P'_j\, y^*$. Observe that $e' = \left(X, \left(R'_j, R_{-j}\right)\right) \notin \mathcal{E}_{\gamma,y^*,Y}$. Thus, $e' \notin E$, $\mathcal{T}^{(\gamma,y,Y,E)}_j(e') = x$, and $\mathcal{T}^{(\gamma,y,Y,E)}_j(e')\; P_j\; \mathcal{T}^{(\gamma,y,Y,E)}_j(e)$. This contradicts *strategy-proofness*.

Now, we prove that if $E = \emptyset$, then $\mathcal{T}^{(\gamma,y,Y,E)}$ is *strategy-proof*. As tail rules satisfy *2-unanimity*, it is enough to prove that agents can not gain from misreporting their preferences in three-agent economies. Observe that since $E = \emptyset$, then for each three-agent economy $e = (X, R) \in \mathcal{E}^{\mathcal{A}}$, $\mathcal{T}^{(\gamma,y,Y,E)}(e)$ can be calculated with an "endowment inheritance table" that depends only on X and not on the preference profile R (Footnote 24.) "Endowment inheritance rules" satisfy strategy-proofness (Pápai [11].) Thus, agents can not gain from misreporting their preferences. □

Example 8. *A Pareto-efficient and consistent rule that is not an MDPEwt rule when* $|\mathcal{A}| = |\mathcal{X}| = 3$. *Let* $\mathcal{A} = \{i, j, k\}$ *and* $\mathcal{X} = \{a, b, c\}$. *Let* φ *be a rule such that* $F^{\varphi} \supseteq h$, *where*

$$h=\left\{\begin{array}{l} i\succ_{ab}j, i\succ_{ab}k, j\succ_{ab}k, i\succ_{cb}j, i\succ_{cb}k, k\succ_{cb}j, j\succ_{ba}i, k\succ_{ba}i, k\succ_{ba}j,\\ i\succ_{ca}j, k\succ_{ca}i, k\succ_{ca}j, j\succ_{ac}i, i\succ_{ac}k, j\succ_{ac}k, j\succ_{bc}i, k\succ_{bc}i, j\succ_{bc}k,\end{array}\right\}.$$

Observe that, $\mathcal{I}(h, E^{\{a,b,c\}}_{\{i,j,k\}}) = h$.[27] *Thus, for each* $e \in \mathcal{E}^{\mathcal{A}}$, $A(h,e) \neq \emptyset$. *For each* $e \in \mathcal{E}^{\mathcal{A}}$, *let* $\varphi(e) \in A(h,e) \neq \emptyset$. *By construction,* φ *is Pareto-efficient and consistent. Moreover, for each* $n, n' \in \mathcal{A}$, $|\{(x,y) \in \mathcal{X}^2 : n \succ^{\varphi}_{xy} n'\}| = |\{(x,y) \in \mathcal{X}^2 : n' \succ^{\varphi}_{xy} n\}|$. *The rule* φ *is not an MDPEwt rule.*

Example 9. *A 2-unanimous and consistent rule that is not an MDPEwt rule. Let* $\mathcal{A} \equiv \{1,2,3\}$ *and* $\mathcal{X} \equiv \{w,x,y,z\}$. *Let* $(\gamma, y, Y, \emptyset)$ *be a tail structure such that* $\gamma(1) \equiv 1$, $\gamma(2) \equiv 2$, $\gamma(3) \equiv 3$, *and* $Y = \{z\}$. *Let* $\widetilde{e} = (y\,P_i\,x\,P_i\,z, z\,P_j\,y\,P_j\,x, x\,P_k\,z\,P_k\,y)$. *We define a rule* φ. *Let* $\varphi_i(\widetilde{e}) \equiv x$, $\varphi_j(\widetilde{e}) \equiv y$, *and* $\varphi_k(\widetilde{e}) \equiv z$. *For each* $e \in \mathcal{E}$ *such that* $e \neq \widetilde{e}$, *let* $\varphi(e) \equiv \mathcal{T}^{(\gamma,y,Y,E)}(e)$. *The rule* φ *is 2-unanimous and consistent, but it is not 3-unanimous. Thus, it is not an MDPEwt rule.*

Example 10. *A Pareto-efficient and 2-consistent rule that is not an MDPEwt rule. Let* $\mathcal{A} \equiv \{1,2,3,4\}$ *and* $\mathcal{X} \equiv \{a,b,c,d\}$. *Let* $(\Pi, \succ) \equiv \{1\} \succ \{2,3,4\}$. *Let* (γ, a, Y, E) *be a tail structure (for the set of potential agents* $\{2,3,4\}$.*) Let* $E' \subseteq \mathcal{E}_{\gamma,a,Y}$ *be such that* $E' \neq E$. *Let* $Q \equiv \mathcal{T}^{(\gamma,a,Y,E)}$ *and* $Q' \equiv \mathcal{T}^{(\gamma,a,Y,E')}$. *We define a rule* φ. *For each* $e \in \mathcal{E}^{\mathcal{A}}$, *let* $\varphi(e) \equiv D^{\left(\Pi,\succ,T^{\emptyset},Q\right)}$, *and for each* $N \in \mathcal{N}$ *such that* $|N| \leq 3$, *and each* $e \in \mathcal{E}^N$, *let* $\varphi(e) \equiv D^{\left(\Pi,\succ,T^{\emptyset},Q'\right)}$. *The rule* φ *is Pareto-efficient and 2-consistent, but it is not 3-consistent. Thus, it is not an MDPEwt rule.*

Computation of $\mathcal{I}(h_0, E^{\{x,y,z,w\}}_{\{i,j,k\}})$ **in Example 7:** Let $h_0 \equiv \{i \succ_{xy} j,\ j \succ_{xy} k,\ j \succ_{xz} i\}$. Table 3 presents the implications of h_0 obtained by analyzing economies in $E^{\{x,y,z,w\}}_{\{i,j,k\}}$.

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Set	Subset h	Test economy e	$\mathcal{I}(h,e)\cap h^c$
h_0	$\{i \succ_{xy} j, j \succ_{xz} i\}$	$(R_i^{xyz}, R_j^{xzy}, R_k^{yxz})$	$k \succ_{yz} i$
$h_1 \equiv h_0 \cup \{k \succ_{yz} i\}$	$\{i \succ_{xy} j, j \succ_{xz} i, j \succ_{xy} k\}$	$(R_i^{xyz}, R_j^{xzy}, R_k^{zxy})$	$k \succ_{zy} j$
$h_2 \equiv h_1 \cup \{k \succ_{zy} j\}$	$\{i \succ_{xy} j, j \succ_{xy} k\}$	$(R_i^{wxy}, R_j^{xyw}, R_k^{xwy})$	$i \succ_{wy} k$
$h_3 \equiv h_2 \cup \{i \succ_{wy} k\}$	$\{i \succ_{wy} k, k \succ_{zy} j\}$	$(R_i^{zwy}, R_j^{zyw}, R_k^{wzy})$	$i \succ_{zy} j$
$h_4 \equiv h_3 \cup \{i \succ_{zy} j\}$	$\{i \succ_{wy} k, k \succ_{zy} j\}$	$(R_i^{wyz}, R_j^{zyw}, R_k^{wzy})$	$i \succ_{wz} k$
$h_5 \equiv h_4 \cup \{i \succ_{wz} k\}$	$\{i \succ_{wy} k, k \succ_{zy} j\}$	$(R_i^{wyz}, R_j^{zwy}, R_k^{wzy})$	$i \succ_{wy} j$
$h_6 \equiv h_5 \cup \{i \succ_{wy} j\}$	$\{i \succ_{zy} j, i \succ_{xy} j, j \succ_{xy} k\}$	$(R_i^{xzy}, R_j^{zxy}, R_k^{xyz})$	$i \succ_{xy} k$
$h_7 \equiv h_6 \cup \{i \succ_{xy} k\}$	$\left\{\begin{array}{c} k \succ_{zy} j, i \succ_{xy} j, \\ i \succ_{xy} k, j \succ_{xy} k \end{array}\right\}$	$(R_i^{xzy}, R_j^{zxy}, R_k^{xzy})$	$i \succ_{xz} k$
$h_8 \equiv h_7 \cup \{i \succ_{xz} k\}$	$\left\{\begin{array}{c} k \succ_{xy} j, i \succ_{xy} j, \\ i \succ_{xy} k, j \succ_{xy} k \end{array}\right\}$	$(R_i^{zxy}, R_j^{zxy}, R_k^{xzy})$	$i \succ_{zx} j$
$h_9 \equiv h_8 \cup \{i \succ_{zx} j\}$	$\{i \succ_{xy} k, j \succ_{xy} k\}$	$(R_i^{xyw}, R_j^{xyw}, R_k^{xwy})$	$i \succ_{xw} k$
$h_{10} \equiv h_9 \cup \{i \succ_{xw} k\}$	$\{i \succ_{wy} j, j \succ_{xy} k\}$	$(R_i^{wyx}, R_j^{wxy}, R_k^{xyw})$	$i \succ_{wx} j$
$h_{11} \equiv h_{10} \cup \{i \succ_{wx} j\}$	$\{j \succ_{xz} i, i \succ_{xz} k, i \succ_{xw} k\}$	$(R_i^{xzw}, R_j^{xzw}, R_k^{xwz})$	$j \succ_{xw} k$
$h_{12} \equiv h_{11} \cup \{j \succ_{xw} k\}$	$\{j \succ_{xz} i, i \succ_{xz} k\}$	$(R_i^{xzw}, R_j^{wxz}, R_k^{xwz})$	$j \succ_{wz} k$
$h_{14} \equiv h_{13} \cup \{j \succ_{wz} k\}$	$\left\{\begin{array}{c} i \succ_{wz} k, j \succ_{xz} i, \\ i \succ_{xz} k, i \succ_{xw} k \end{array}\right\}$	$(R_i^{xwz}, R_j^{xzw}, R_k^{xwz})$	$j \succ_{xw} i$
$h_{16} \equiv h_{15} \cup \{j \succ_{xw} i\}$	$\{j \succ_{xw} i, i \succ_{xw} k\}$	$(R_i^{xwz}, R_j^{zxw}, R_k^{xzw})$	$j \succ_{zw} k$
$h_{16} \equiv h_{15} \cup \{j \succ_{zw} k\}$	$\{i \succ_{zx} j, i \succ_{xw} k, j \succ_{xw} k\}$	$(R_i^{zxw}, R_j^{zxw}, R_k^{xzw})$	$i \succ_{zw} k$
$h_{17} \equiv h_{16} \cup \{i \succ_{zw} k\}$	$\{i \succ_{wx} j, i \succ_{xz} k\}$	$(R_i^{wxz}, R_j^{wzx}, R_k^{xzw})$	$i \succ_{wz} j$
$h_{18} \equiv h_{17} \cup \{i \succ_{wz} j\}$	$\{k \succ_{yz} i, j \succ_{xz} k\}$	$(R_i^{yzw}, R_j^{ywz}, R_k^{wyz})$	$j \succ_{yz} i$
$h_{19} \equiv h_{18} \cup \{j \succ_{yz} i\}$	$\{i \succ_{zw} k, k \succ_{zy} j, j \succ_{zw} k\}$	$(R_i^{yzw}, R_j^{zyw}, R_k^{zyw})$	$i \succ_{yw} k$
$h_{20} \equiv h_{19} \cup \{i \succ_{yw} k\}$	$\{i \succ_{yz} j, k \succ_{yz} i, i \succ_{yw} k\}$	$(R_i^{ywz}, R_j^{yzw}, R_k^{yzw})$	$j \succ_{yz} k$
$h_{21} \equiv h_{20} \cup \{j \succ_{yz} k\}$	$\{i \succ_{wz} j, k \succ_{yz} i, j \succ_{yz} k\}$	$(R_i^{ywz}, R_j^{wyz}, R_k^{yzw})$	$\mathcal{X}^2 \times \mathcal{A}^2$

Table 3: Computation of $\mathcal{I}(h_0, E_{\{i,j,k\}}^{\{x,y,z,w\}})$ in Example 7.