Pricing of Durable Network Goods under Dynamic Coordination Failure*

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March 12, 2007

Abstract

In markets of durable goods with network externalities, consumers make purchase decisions over time given a product price, which amounts to playing a dynamic coordination game. The game typically has multiple equilibria with different degrees of inefficient coordination failure. The present paper fully characterizes, in the context of monopoly, the set of profits sustained by an equilibrium of a setting where the firm first commits to a price and then the consumers play the dynamic coordination game. Based on the idea that the monopolist prefers selling strategy which favorably changes the equilibrium profit set, we apply the result to such issues as relevance of tie-in sales. We also show that the firm suffers if it cannot commit to its price.

JEL Classification: C72, C73, D42, L12.
Keywords: Dynamic coordination games, monopoly pricing, tie-in sales.

*This project stems from stimulating conversations with Tatsuhiro Shichijo. The author also thanks comments from Chiaki Hara, Atsushi Kajii, Kazuharu Kiyono, Daisuke Oyama, In-Uck Park, Takashi Shimizu, Jun Xue, and seminar participants at Kyoto, Cambridge, Hitotsubashi, Hokkaido, Yonsei, Waseda and the CTW Workshop. Financial supports from the Grants-in-Aids for Scientific Research (16203011, 17730129) and the 21st Century COE Research, are gratefully acknowledged.

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1 Introduction

Firms selling network goods, namely goods whose intrinsic values depend on the size of customers, face a coordination problem on the consumer side. For goods or services like computers or telephony (whether cellular or not), it is natural to assume that the value of the products is an increasing function of the number of adopters. Consumers who anticipate that other consumers also buy are more willing to purchase the goods. In contrast, when consumers believe that others would not buy, then they are reluctant to buy, because its "stand-alone" value (the value consumers can enjoy aside from all network externalities) may not be large enough to recoup the cost of purchase. This is a typical coordination failure, causing firms producing high-quality network goods to sell them at a relatively low price or to fail to sell them at all.

If, in addition, the goods are durable, the game played by the consumers becomes a dynamic coordination game introduced by Gale (1995). In this type of games, consumers have an incentive to sacrifice themselves by buying early, in an effort to establish a critical mass and speed up coordination. Consequently, as Gale (1995) pointed out, in any equilibrium of the dynamic coordination game played by patient consumers, full coordination is achieved sooner or later. This mitigates the coordination problem, but multiple equilibria with varying timing of adoption still exist. Without knowing which equilibrium is to be played, the firm is unable to design its pricing policy.

This paper addresses the problem a firm faces under those circumstances. In particular, we consider a simple case of monopoly in a market with identical consumers, and assume that the monopolistic firm has no idea as to which equilibrium is to be played by the consumers, given the price it chooses. Thus our first task is to determine the whole set of possibilities that can arise in equilibrium, and more importantly, the profit levels of the monopolist brought about by those equilibria. Hence we first characterize the whole set of the monopolist's profits sustained by an equilibrium of the dynamic coordination game.[1]

Since the equilibrium profit set is shown to be a closed (possibly degenerate) interval, we next characterize its maximum and minimum. The maximum profit is, naturally, the price times the number of consumers (unless the price is too high). This corresponds to an equilibrium where consumers immediately buy. The equilibrium exhibits no coordination failure at all. Less obvious is the minimum profit. If the price is too low, then even stand-alone values of the network goods are so large as to recoup the cost, so buying now cannot be suboptimal. In this case, the minimum equals the maximum, and the interval is degenerate. In contrast, if the price is too high, then there exists an equilibrium where no consumer ever purchases. This is the worst possible coordination failure for the monopolist, and the minimum profit is zero. For intermediate prices, the worst profit obtains under an equilibrium where consumers delay purchases. In this case, the worst profit depends on all parameters of the model, including both full and intermediate values of the network goods, as well as discount

[1] To obtain a clean result, we consider a continuous-time version of the dynamic coordination games where the consumers can decide to buy any time. However, their purchase decisions are observed with a lag, during which they are observed as if they did not buy. The observation lag plays the same role as the period length in Gale's (1995) discrete time model, but allows us to compute the purchase timing under equilibrium without suffering from an integer problem.

rates. In particular, the worst profit is increasing in the intermediate values, because higher those values speed up the coordination process. Also it is decreasing in the discount rate of consumers, and converges to the maximum if the rate goes to zero (complete patience). That is, the loss associated with coordination failure vanishes if the consumers are sufficiently patient.

Given the multiplicity of equilibria, we allow a possibility that the monopolist does not have a theory which selects one equilibrium given the price it charges. Then a modest thing we can say about the firm would be that it prefers a situation where the profit set interval "increases;" namely, the case where its minimum and maximum weakly increases, and at least one of them increases. Building on that idea, we explore implications of the result on the firm's pricing strategies. As we mentioned above, the monopolist prefers policies which increases any kind of value of the network goods. Therefore, if the firm is in a position to make costly initial investments in qualities of its products, then our theory predicts that it would optimally invest so that the worst profit improves. Also the result is related to whether it pays or not to bundle two network goods, over both of which the firm has monopoly power. In Subsection 4.2, we show that under certain conditions separate sales yield a greater worst profit than tie-in sales. One special case is bundling of network and nonnetwork goods. Thus the result predicts that the network goods contain nonnetwork additional values (for instances, clock and calculator functions of a cellular phone) only when the seller does not have strong monopoly power over those values.

We can derive stronger implications of our result, if we explicitly introduce an equilibrium selection criterion. Since our analysis puts much more emphasis on dynamic coordination failure, one possible criterion, though quite extreme, is that the worst equilibrium is always played. In Subsection 4.3, we employ the criterion and compute the optimal price of the monopolist which maximizes the worst profit. Such a "maxmin" behavior might be a good approximation of behavior of a pessimistic firm. We show that the optimal price exists if consumers are sufficiently patient, and converges to the full aggregate value of the network goods as the discount rate goes to zero. Also the optimal price is increasing in both full and intermediate values of the network goods.

The analysis of pricing of network goods dates back to Rohlfs (1974), and is extended by Katz and Shapiro (1985, 1986) to the case of duopoly and by Cabral, Salant, and Woroch (1999) to the case of dynamic monopoly with incomplete information. Cabral, Salant, and Woroch (1999) also deal with the case of complete information, but conclude that the complete information case is not tractable because of the multiplicity.[2] In this paper, on the contrary, we argue that models with complete information still yield meaningful predictions if we treat multiplicity as it is, or if we impose selection criteria on it (like the analysis in Subsection 4.3).

The dynamic coordination game the consumers play in our model is first introduced

[2] Cabral, Salant, and Woroch (1999) even consider equilibria in which a particular pricing pattern is supported because any deviation by the monopolist is punished by consumers switching to a bad equilibrium. In other words, the monopolist adopts a selection criterion which punishes itself in case of deviation. Our analysis does not share that feature, because (i) we first analyze the model with no selection criterion, and (ii) even when we need a criterion, we do not employ one which rewards or punishes the firm depending on the price. In other words, in our criterion the firm is always punished.

by Gale (1995), who already pointed out multiplicity of equilibria. However, Gale (1995) does not characterize the whole set of equilibria, or does not solve for the worst equilibrium. Indeed, the whole equilibrium set of the dynamic coordination game is hard to obtain, because there are various patterns of behavior consistent with an equilibrium.[3] Nevertheless, the corresponding profit of the firm is much easier to compute, and one contribution of this paper is to give a full characterization of equilibria in dynamic coordination games from a different perspective.

The remainder of the paper is organized as follows. The next section describes the model. Section 3 provides our main result, a characterization of the set of the firm's profit given by an equilibrium of the dynamic coordination game. Section 4 deals with applications of the result of Section 3; the issues of optimal design of network goods and tie-in sales. We also introduce an equilibrium selection argument that the worst equilibrium given the price is always played, and analyze the optimal price of the monopolist. Section 5 comments on extensions to heterogeneous discounting and the firm's inability to commit to its price. Section 6 concludes.

2 Model

There is a single market of network goods, consisting of a monopolistic firm and $N \geq 2$ identical consumers. The time line is a continuum $[0, \infty)$. At time 0, the monopolist chooses its price p. Here we assume that the firm commits to a constant price over time.[4] The production cost is assumed to be zero.

The consumers have unit demands for the network goods. Given the price, they decide when to make an irreversible decision to buy it. Thus once a consumer purchases the goods, she has no further decision opportunity and can be regarded as quitting from the game. In the course of play, the consumers who have yet to buy acquire information about the other consumers' purchase decisions, but the information comes with a lag. More concretely, if a consumer buys at some time t, then the other consumers learn that at time $t+\tau$, where $\tau > 0$. Hence by time $t+\tau$, the consumer is seen as if she had yet to buy. Thus at time t, the history of a consumer is other consumers' all purchase decisions made by time $t-\tau$. Her strategy is a function which maps a history at any time to a probability of purchase at that time.

The value of the network goods depends on the number of consumers who have already purchased them. Recall that all consumers are identical, and we let z_k be its instantaneous utility for each consumer if there are exactly k consumers, including herself, who have bought by that time. We assume $z_k > 0$ for any k, and z_k is increasing in k with $z_N > z_1$.[5] We call z_N the full value, other z_k's the intermediate values, and sometimes z_1 the stand-alone value. We assume that the consumers and the firm have

[3] See Appendix B for some details.

[4] In Subsection 5.2, we briefly comment on the case where the firm may change the price over time, depending on past sales. An alternative approach is to allow the firm to commit to a time-varying price schedule. See Park (2004) and Shichijo and Nakayama (2004) for alternative approaches.

[5] It may seem that this specification is inconsistent with the assumption of information lag, because the consumers can learn the number of adopters immediately by checking their instantaneous payoff level. This is not the case, however, because only adopters can learn the number of adopters from their payoffs. Since adopters virtually exit from the game, this additional information is useless.

the same discount rates, denoted by $r > 0$.[6] The overall utility of a consumer who buys the goods at time t at the price p is:

$$E\Big[\int_t^\infty e^{-rs} z_{k(s)} ds\Big] - e^{-rt}p,$$

where the expectation is taken with respect to other consumers' mixed strategies, and where $k(s)$ is the number of adopters at time s. The profit of the firm when each consumer i made purchase in time t_i at price p is $p \cdot \sum_{i=1}^N \exp(-rt_i)$ (regard $t_i = \infty$ if consumer i does not buy it). We denote this dynamic game played by the consumers under the price p by $\Gamma(p)$. It is a continuous-time version of the dynamic coordination game by Gale (1995), with a lag in information transmission. Since no rational consumer would buy if $p > z_N/r$, we hereafter assume $p \leq z_N/r$.

Let p and $\Gamma(p)$ be given. A perfect Bayesian equilibrium of $\Gamma(p)$ is a profile of strategies such that given any history, the continuation strategy of each consumer who has yet to buy is optimal, given a consistent belief on the other consumers' future plays. For any p, $\Gamma(p)$ has a very simple perfect Bayesian equilibrium such that at any history each consumer immediately buys if she has yet to do so. This strategy profile has the path where all consumers buy at time 0. At any history, since all other consumers are believed to have bought or be about to buy now, each consumer's continuation value is $(z_N/r) - p \geq 0$, which is her best possible payoff in $\Gamma(p)$. Hence the profile is indeed a perfect Bayesian equilibrium. Since all goods sell immediately, it is also the best scenario for the firm.

To conclude, for any $p \leq z_N/r$, the best equilibrium profit of the firm in $\Gamma(p)$ is Np. In the next section, we derive other profit levels sustained by an equilibrium.

3 Characterization of the Equilibrium Profits

In this section, we determine the whole set of equilibrium profits for the monopolist. To this end, the following result is helpful, which describes a relationship between consumers' utilities and the firm's profit.

Lemma 1. *Let p be given. Fix $u \in (0, (z_N/r) - p]$, and choose any strategy profile of $\Gamma(p)$ where the average utilities of the consumers are u. Then if we set a number $T \geq 0$ so that*

$$u = e^{-rT}\left(\frac{z_N}{r} - p\right), \tag{1}$$

the firm's profit under the profile is no less than $Np \cdot \exp(-rT)$.

Proof. Let $p_k(t)$ be the probability that, given the profile, exactly k consumers

[6] Subsection 5.1 discusses the case with heterogeneous discount rates.

have bought by time t. Then by the definition of u, we have:

$$
\begin{aligned}
Nu &= \int_0^\infty \Big\{\sum_{k=1}^N p_k(t)k(z_k - rp)\Big\} e^{-rt}dt \\
&\le \int_0^\infty \Big\{\sum_{k=1}^N p_k(t)k(z_N - rp)\Big\} e^{-rt}dt,
\end{aligned}
$$

where the inequality follows from monotonicity of z_k's. Substituting (1), we obtain:

$$
\frac{N}{r}e^{-rT} \le \int_0^\infty \Big\{\sum_{k=1}^N p_k(t)k\Big\} e^{-rt}dt.
$$

Hence the firm's profit $\hat{\pi}$ under the profile satisfies

$$
\hat{\pi} = \int_0^\infty \Big(\sum_{k=1}^N p_k(t)rpk\Big) e^{-rt}dt \ge Np \cdot e^{-rT},
$$

which completes the proof. Q.E.D.

Lemma 1 implies that the more utilities the consumers receive, the more profit the firm is guaranteed. This is natural, because consumers get positive utilities only if they make purchases, which surely benefits the firm. Indeed, the minimum profit presented in Lemma 1, $Np \cdot \exp(-rT)$, is easy to interpret. For consumers, the most efficient way to earn utilities is to buy simultaneously. Given u, the most efficient way to get the average utility of u is to buy simultaneously at time T. This implies that the consumers coordinate most tardily, and therefore this is the worst scenario to the firm. Note that the result holds independently whether the strategy profile in question is an equilibrium or not.

Now we are ready to characterize the equilibrium profit set of $\Gamma(p)$ for any given p. We first consider the case of $N = 2$ and characterize a perfect Bayesian equilibrium which gives the firm the smallest profit. Then we derive the whole equilibrium profit set, and finally consider the case with $N \ge 3$.

Proposition 1. *Assume $N = 2$. Define $\delta = \exp(-r\tau)$ and*

$$
\begin{aligned}
\underline{p} &= \frac{z_1}{r}, \\
\bar{p} &= \frac{(1-\delta)z_1 + \delta z_2}{r}.
\end{aligned}
$$

(i) If $p < \underline{p}$, then the perfect Bayesian equilibrium outcome of $\Gamma(p)$ is unique, which is such that all consumers buy immediately at time 0.

(ii) If $p \in [\underline{p}, \bar{p})$, then there exists a perfect Bayesian equilibrium which gives the smallest profit to the firm among all perfect Bayesian equilibria of $\Gamma(p)$, with the

following path; all consumers buy at time T, where T is such that

$$\bar{p} - p = e^{-rT}\left(\frac{z_2}{r} - p\right). \tag{2}$$

(iii) If $p \geq \bar{p}$, then there exists a perfect Bayesian equilibrium of $\Gamma(p)$ with a path such that no consumer buys the goods.

Proof. Let us start with the case $p < \underline{p}$. Fix a perfect Bayesian equilibrium of $\Gamma(p)$ and its strategy of consumer $j = 1, 2$, denoted by σ_j. By sequential rationality, σ_j must prescribe purchase at any history when consumer j knows that i has bought. Suppose consumer i buys immediately against σ_j. None of her strategy induces consumer j to buy earlier than buying now, because (i) if σ_j is such that j buys by time τ with probability 1, i's strategy does not affect j's purchase timing at all, and (ii) if σ_j prescribes with a positive probability to wait until time τ, buying now induces j to buy at time τ with that probability. Moreover, buying now gives consumer i the instantaneous payoff of $z_1 - rp$ until j buys, which is her best possible payoff given that j has yet to buy. Also it gives the instantenous payoff of $z_2 - rp$ after j buys, which is her best possible payoff given that j has bought. Consequently, buying now is a best response against σ_j. Also it is a unique best response because delaying purchase causes consumer i to lose at least instanteneous payoffs of $z_1 - rp > 0$ by the time she buys. Hence the perfect Bayesian equilibrium of $\Gamma(p)$ must consist of immediate purchases by both consumers, which proves (i).

Next, suppose $p \in [\underline{p}, \bar{p})$. Since $z_2 > rp$, in any perfect Bayesian equilibrium of $\Gamma(p)$, any consumer who learned that the other consumer has bought immediately buys. This means that given any perfect Bayesian equilibrium, a consumer can guarantee the payoff of

$$\int_0^\tau e^{-rs}(z_1 - rp)ds + \int_\tau^\infty e^{-rs}(z_2 - rp)ds = \bar{p} - p,$$

by buying at time 0. Thus any perfect Bayesian equilibrium gives any consumer at least $\bar{p} - p$. Therefore by Lemma 1, if there exists a perfect Bayesian equilibrium where all consumers buy at time T, it is clearly the worst equilibrium for the firm.

Consider the following strategy profile of $\Gamma(p)$. Each consumer immediately buys if either (a) she is at time $t \geq T$, where T is given by (2), or (b) she learned that the other consumer had bought. Otherwise, she does not buy. Under this profile, all consumers buy at time T. To see that the profile is a perfect Bayesian equilibrium, first note that it is always optimal for a consumer to buy if she knows that the other consumer has already bought. Thus sequential rationality for the case (b) is trivial. For the case (a), a consistent belief at such a history is that the other consumer has already bought or is about to buy now. In any case, it is optimal to buy now. If a consumer is at time $t < T$ and has not observed that the other consumer has bought, then the only consistent belief given the profile is that he has yet to buy. If $t \geq T - \tau$, then her purchase decision does not affect the other consumer's play (because he anyway buys at time T, before knowing her decision to buy now), so it is optimal to wait until time T because $z_1 \leq rp$. Otherwise, her current purchase induces the other consumer to buy at time $t+\tau$, but her current value payoff is

$$\int_0^{\tau} e^{-rs}(z_1 - rp)ds + \int_{\tau}^{\infty} e^{-rs}(z_2 - rp)ds = \bar{p} - p \leq e^{-r(T-t)}\left(\frac{z_2}{r} - p\right), \tag{3}$$

where the inequality is by (2). Since the RHS of (3) is her payoff if she conforms to the strategy, sequential rationality follows for that case, too. Hence the profile is a perfect Bayesian equilibrium. This completes the proof of part (ii).

Finally, suppose $p \geq \bar{p}$, and consider the following strategy profile. Each consumer immediately buys if she learned that the other consumer had bought. Otherwise, she does not buy. Under this profile, all consumers do not buy forever. We show that the profile is a perfect Bayesian equilibrium. Again, the histories at which the other consumer has bought are trivial. So consider the case where a consumer has not observed that the other consumer has bought. Given the other consumer's strategy, the only consistent belief is that he has yet to buy and will not buy until he knows that she bought. Then the consumer's payoff if she buys now is $\bar{p} - p \leq 0$. Since zero is the payoff of not buying forever, her continuation play given the profile is sequentially rational, which proves (iii). Q.E.D.

The intuition of Proposition 1 is clear. If $p < \underline{p}$, then p is smaller than the stand-alone value. Therefore, buying now is "dominant," denying a possibility of other equilibria. If $p \geq \underline{p}$, then buying now is no longer dominant, but doing so guarantees that the other consumer buys by time τ. Consequently, the consumer can secure the payoff of $\bar{p} - p$. If $\bar{p} - p > 0$, then as Lemma 1 indicates, the worst scenario for the firm is that they simultaneously buy at time T. This is indeed an equilibrium, so that it gives the worst profit. If $\bar{p} - p \leq 0$, then perpetual coordination failure is supported by an equilibrium, because buying now cannot secure a greater payoff.

To sum up, the firm's worst profit is:

$$\pi(p) = \begin{cases} Np & \text{if } p < \underline{p}, \\ Np \cdot e^{-rT} & \text{if } p \in [\underline{p}, \bar{p}), \\ 0 & \text{if } p \geq \bar{p}. \end{cases} \tag{4}$$

Note that π exhibits discontinuity at $\underline{p}$ and $\bar{p}$.

If $p < \underline{p}$, $\pi(p)$ equals the best profit we derived in Section 2. Hence the equilibrium profit set is a singleton. If $p \in [\underline{p}, \bar{p})$, then for any $T' < T$, there exists a perfect Bayesian equilibrium such that both consumers simultaneously buy at time T'. Hence any $\pi' \in [\pi(p), Np]$ is achieved as an equilibrium profit. Finally, if $p \geq \bar{p}$, then for any $T' \geq 0$, there exists a perfect Bayesian equilibrium such that both consumers simultaneously buy at time T'. Therefore, any $\pi' \in [\pi(p), Np]$ is achieved as an equilibrium profit. Hence we have established the following result.

Proposition 2. *For any p, the set of profits sustained by an equilibrium of $\Gamma(p)$ is the interval $[\pi(p), Np]$, where π is defined by (4).*

Next, we consider the case of $N \geq 3$. Fix $p \leq z_N/r$, and let $K \geq 0$ be a (unique) integer such that $z_K/r \leq p < z_{K+1}/r$, where we set $z_0 = 0$ and $z_{N+1} = \infty$. For

$k = 0, 1, \cdots, N$, we inductively define a sequence of numbers $(T_k)_{k=0}^N$ as follows:

$$T_k = 0 \quad \text{if } k \geq K,$$
$$\frac{(1 - \delta e^{-rT_k})z_k + \delta e^{-rT_k} \cdot z_N}{r} - p = e^{-rT_{k-1}} \left(\frac{z_N}{r} - p \right) \quad \text{if } k \leq K. \tag{5}$$

T_{k-1} in the RHS of (5) does not exist if its LHS is nonpositive. If that happens, define $T_{k-1} = \infty$, and set $T_{k'} = \infty$ for any $k' < k-1$. Also, if $K = N$ (namely, $p = z_N/r$), set $T_k = \infty$ for any $k \leq N-1$. It is easily seen that $(T_k)_{k=0}^N$ is strictly decreasing unless they hit either zero or infinity. That is, if $T_k \neq 0$ and $T_{k-1} \neq \infty$, then $T_k < T_{k-1}$.

The sequence $(T_k)_{k=0}^N$ is important because it determines the bound on the continuation payoff at any history of any perfect Bayesian equilibrium, as the following result demonstrates.

Lemma 2. *Fix p, and let $(T_k)_{k=0}^N$ be defined as above. For any perfect Bayesian equilibrium of $\Gamma(p)$ and for any history of a consumer at time t at which she learned that k consumers have already bought, her current value continuation payoff is at least $\exp(-rT_k)[(z_N/r) - p]$.*

Proof. See Appendix A.

Remark 1. If $T_k = \infty$, then the lemma puts no restriction on the continuation payoff given the history.

The next proposition solves the equilibrium profit set for a general number of consumers.

Proposition 3. *For any given p, the equilibrium profit set of the firm in $\Gamma(p)$ is the interval $[Np \cdot \exp(-rT_0), Np]$ if $T_0 \neq \infty$, and $[0, Np]$ otherwise.*

Remark 2. Proposition 3 is a generalization of Proposition 2. Indeed, if $N = 2$, (i) $T_0 = 0$ if $p < \underline{p}$ (because $K = 0$), (ii) T_0 equals to T defined by (2) if $p \in [\underline{p}, \bar{p})$, and (iii) $T_0 = \infty$ if $p \geq \bar{p}$.

Proof. First, suppose $T_0 \neq \infty$. Then by Lemma 2, in any perfect Bayesian equilibrium of $\Gamma(p)$, each consumer is guaranteed a payoff of $\exp(-rT_0)[(z_N/r) - p]$. Therefore Lemma 1 implies that if there exists a perfect Bayesian equilibrium where all consumers simultaneously buy at time T_0, then the equilibrium gives the smallest profit to the firm.

Let us consider the following strategy profile of $\Gamma(p)$. At any history of a consumer at time t where $k \geq 0$ consumers are observed to have bought, let t' be the time when the k-th consumer has made purchase. If $k = 0$, let $t' = 0$. Then she buys if $t \geq t' + T_k$ and she does not buy otherwise.

The profile has a path where all consumers simultaneously buy at time T_0. We show that the profile is a perfect Bayesian equilibrium. Consider first a history at which a consumer is prescribed to buy. At such a history, all consumers who have yet to buy are at a history they are prescribed to buy now. Hence buying now is a best response at that history. Next, consider a history at which the consumer is prescribed not to buy. Let k be the number of consumers who are observed to have bought, and t' be the time when the k-th consumer had bought. Then all consumers who have yet to buy are at a history with the same k and t'. Therefore, the continuation play is that all remaining consumers simultaneously buy at time $t'+T_k$, with the continuation payoff no less than $\exp(-rT_k)[(z_N/r)-p]$. In contrast, if the consumer deviates and buys now, we have two cases to consider. First, if $t'+T_k-t \leq \tau$, then her current purchase does not affect the continuation play. Thus it is not optimal to buy now, because $T_k > 0$ implies that $z_k/r < p$. Second, if $t'+T_k-t > \tau$, then at time $t+\tau$, the other consumers confront a history with $k+1$ and $t' = t+\tau$. Therefore, their continuation play is that they simultaneously buy at time $t+\tau+T_{k+1}$. As a result, the continuation payoff of a consumer who buys now is equal to $\exp(-rT_k)[(z_N/r)-p]$ by (5). Therefore the deviation does not pay, and we have proved that the strategy profile is a perfect Bayesian equilibrium.

For any $T \leq T_0$, there exists a perfect Bayesian equilibrium where all consumers simultaneously buy at time T: simply modify the above profile so that T_0 is replaced with T. This demonstrates that the equilibrium profit set of $\Gamma(p)$ is $[Np \cdot \exp(-rT_0), Np]$.

Next, suppose $T_0 = \infty$. Modify the above profile so that each consumer does not buy at any history with k such that $T_k = \infty$. The modified profile has the path where no consumer buys. We show that the profile is a perfect Bayesian equilibrium. Any history with k such that $T_k \neq \infty$ can be dealt with in the same way as above, so let us consider the remaining histories. First consider the case where $T_k = \infty$ and $T_{k+1} \neq \infty$. A similar argument shows that at such a history, if a consumer buys now, the continuation payoff is the LHS of (5). Since $T_k = \infty$, the payoff is negative. Since the continuation strategy profile prescribes no future purchase at all and therefore gives the continuation payoff of zero, the deviation does not pay. If $T_k = T_{k+1} = \infty$, then the continuation play when she buys at the history is that no further purchase occurs. Hence the continuation payoff is $(z_{k+1}/r)-p < 0$, where the inequality follows because $k+1 < K$. The deviation does not pay at this history either, so that the profile is a perfect Bayesian equilibrium. Since the firm's profit is zero, this is clearly the worst profit.

For any $T \geq 0$, there exists a perfect Bayesian equilibrium where all consumers simultaneously buy at time T: further modify the profile so that at history with $k = 0$, the consumers buy if and only if $t \geq T$. This demonstrates that the equilibrium profit set in this case is $[0, Np]$. Q.E.D.

Using (5), we can solve T_0 for $\exp(-rT_0)$, when it is not infinity:

$$e^{-rT_0} = 1 - \frac{1-\delta}{\delta} \sum_{k=1}^{K} \left\{ \prod_{l=1}^{k} \left(\delta \frac{z_N - z_l}{z_N - rp} \right) \right\}. \tag{6}$$

(6) reveals how the worst profit of the firm, $Np \cdot \exp(-rT_0)$, depends on parameters of the model. It is easily seen that it strictly increases with z_k for any k, and strictly decreases with r and τ (note that δ reflects both r and τ, as $\delta = \exp(-r\tau)$). Those results are all intuitive. Higher values (whether full or intermediate) make it more attractive for consumers to make early purchases, so that the dynamic coordination

speeds up. More implications of the result are provided in the next section. Less patience (large r) and long information lag (large τ) decrease the worst profit, because those factors make early purchases less attractive.

Also (6) shows that the worst profit converges to the best profit (Np) as $\delta \to 1$. δ can be made large by two ways: (i) for a fixed discount rate r, having a shorter information lag τ, and (ii) for a fixed information lag, having a smaller discount rate. In either case, if $\delta \to 1$, the consumers almost immediately buy in any equilibrium, and the firm loses little even from the worst equilibrium. In this situation, the firm does not suffer from uncertainty as to which equilibrium is played, and therefore would behave as if it does not care about it.

We conclude this section by pointing out that while we fully characterize the equilibrium profit of the firm, the whole set of perfect Bayesian equilibria is much more complicated. We have shown that any equilibrium profit is sustained by a simple, symmetric pure strategy equilibrium, but $\Gamma(p)$ generally admits other types of equilibria. It may have asymmetric equilibria and mixed strategy equilibria. In Appendix B, we provide some examples.

4 Applications

In this section, we present applications of the results in Section 3 to the firm's marketing strategies. We first consider the issues of design of network goods and tie-in sales of multiple network goods. We also discuss how our result can be strengthened if we combine it with an equilibrium selection criterion, which selects one equilibrium of each $\Gamma(p)$. More specifically, we employ one extreme criterion which always selects the *worst* equilibrium to the firm in terms of its profit. Then we solve a *maxmin price*, which is the price maximizing the worst profit, and examine its properties.

4.1 Design of Network Goods

It is natural to imagine that the values of the network goods are determined by both the demand structure and the overall configuration of the products. Let us concentrate on the latter, and assume that the value profile $(z_k)_{k=1}^N$ reflects the firm's initial investment in its products. Then, given the structure of the equilibria we characterized in Section 3, how does the monopolist design its network goods?

First of all, we must clarify what constitutes the values of network goods. Arguably, they consist of connection values, which are strictly increasing in the number of adopters, and stand-alone values, which consumers can enjoy even when there are no other adopters. This view suggests the following relationship between the firm's investments and qualities. The firm chooses two variables $\alpha \geq 0$ and $\beta \geq 0$ as its investments, which correspondingly yield the value profile of:

$$z_k = \alpha y_k + \beta, \quad k = 1, 2, \cdots, N, \tag{7}$$

where $(y_k)_{k=1}^N$ are exogenously given base network values of the products.

α denotes the firm's investment in connection values. Examples are offering clearer telephone conversations, more entertaining softwares, and so forth. β denotes investment in stand-alone values (softwares for individual use, for instance). Given the cost function of those investments, the firm would optimally choose α and β, and design the qualities.

The following result shows that the gross benefit of those investments is always positive, in the sense that they improve the worst profit for any fixed p.

Proposition 4. *Fix $\alpha > 0$, $\beta > 0$ and $p > \beta/r$. Then the worst equilibrium profit of $\Gamma(p)$ increases with a local increase of α and β.*

Proof. We have three cases to consider. If the worst profit equals the best profit, we initially have $p < z_1/r$. Hence an increase of α and β keeps the inequality, so that the worst profit remains to be Np. Next, if the worst profit is zero, then the change in α and β cannot decrease it furthermore. Thus the claim trivially follows.

Finally, consider the case where the worst profit is neither zero or Np. Then by (6), the worst profit can be computed as:

$$Np\left[1 - \frac{1-\delta}{\delta}\sum_{k=1}^{K}\left\{\prod_{l=1}^{k}\left(\delta\frac{y_N - y_l}{y_N + \frac{\beta - rp}{\alpha}}\right)\right\}\right],$$

where K is defined as in (5). Clearly it is increasing both in α and β (note that $p > \beta/r$). Q.E.D.

We emphasize that the result should not be misunderstood. Proposition 4 does *not* say that the profit increases because of increased price control; the price is held fixed. It rather states that the worst profit improves because the consumers are now more willing to buy early, and therefore dynamic coordination failure mitigates.

However, this application is more interesting if we assume that the firm would change its price optimally, when its investments increase the values of its products. For such analysis, we need theory as to how the firm chooses its price. Hence we go back to this issue after we introduce the notion of maxmin price in Subsection 4.3.

4.2 Tie-in Sales

Assume that the firm establishes monopoly over two network goods, whose value profiles are $(y_k)_{k=1}^N$ and $(z_k)_{k=1}^N$. The monopolist has two alternatives: to sell those goods separately or to engage in tie-in sales. The latter alternative amounts to selling network goods with the value profile $(y_k + z_k)_{k=1}^N$.

The following result shows that under certain conditions, tie-in sales result in a smaller worst profit than the case where the goods are sold separately under some prices summing up to the original price of the tied-in network goods. Since the best profit is unchanged, we conclude that tie-in sales are an inferior strategy under the conditions.

Definition. Two network goods $(y_k)_{k=1}^N$ and $(z_k)_{k=1}^N$ are *affine* if there exist $\alpha \geq 0$ and β such that:

$$y_k = \alpha z_k + \beta, \quad k = 1, 2, \cdots, N. \tag{8}$$

Two network goods $(y_k)_{k=1}^N$ and $(z_k)_{k=1}^N$ are *homogeneous* if we have $\beta = 0$ in (8).

Proposition 5. *Assume two network goods with value profiles $(y_k)_{k=1}^N$ and $(z_k)_{k=1}^N$ are affine. Fix p, and let π^T be the worst equilibrium profit of the firm in the game where the two network goods are sold tied-in under the price p. Then there exist a pair of prices p_Y and p_Z such that the following holds.*

(i) $p_Y + p_Z = p$.

(ii) Let π^X $(X = Y, Z)$ be the worst equilibrium profit in the game where $(X_k)_{k=1}^N$ is sold at price p_X. Then we have $\pi^Y + \pi^Z \geq \pi^T$.

(iii) If the network goods are not homogenous and if $0 < \pi^T < Np$, then $\pi^Y + \pi^Z > \pi^T$.

Remark 3. We conjecture that Proposition 5 holds under a much weaker assumption than affinity.

Proof. Without loss of generality, we may assume that $\beta \geq 0$ in (8).

Let us fix p and the corresponding K in (5). If $T_0 = \infty$, then $\pi^T = 0$ so that any separate sales with $p_Y + p_Z = p$ cannot result in smaller aggregate profits. Hence the claim holds. If $T_0 = 0$, then $\pi^T = Np$ and we have $p < (y_1 + z_1)/r$. Therefore if we set

$$p_X = \frac{x_1 p}{y_1 + z_1}, \quad X = Y, Z,$$

we obtain $\pi^X = Np_X$ and therefore $\pi^Y + \pi^Z = \pi^T$, which proves the claim. Note that in both cases (iii) is irrelevant.

Suppose $0 < T_0 < \infty$, which implies $0 < \pi^T < Np$. Define two prices $p_Y(\eta)$ and $p_Z(\eta)$ as:

$$p_Y(\eta) = \frac{\alpha r p + \beta}{(1+\alpha)r} - \eta,$$
$$p_Z(\eta) = \frac{rp - \beta}{(1+\alpha)r} + \eta.$$

It is immediately seen that $p^Y(\eta) + p^Z(\eta) = p$ for any η.

Let $\hat{p}_X = p_X(0)$ for $X = Y, Z$. Note that $X_K/r \leq \hat{p}_X < X_{K+1}/r$ for $X = Y, Z$. Simple computation yields:

$$\pi^T = \sum_{X=Y,Z} \hat{p}_X \left[1 - \frac{1-\delta}{\delta} \sum_{k=1}^{K} \left\{ \prod_{l=1}^{k} \left(\delta \frac{X_N - X_l}{X_N - r\hat{p}_X} \right) \right\} \right]. \tag{9}$$

Hence (ii) holds.

To prove (iii), suppose the two goods are not homogenous; namely, $\beta > 0$. We first consider the case $(y_K + z_K)/r < p$. This implies $X_K/r < \hat{p}_X$ for $X = Y, Z$. Replace $\hat{p}_X$'s with $p_X(\eta)$'s in the RHS of (9), differentiate it with respect to η, and then evaluate it at $\eta = 0$. This produces:

$$\frac{1-\delta}{\delta}\Bigg[\sum_{k=1}^{K}\Bigg\{\prod_{l=1}^{k}\Bigg(\delta\frac{y_N - y_l}{y_N - r\hat{p}_Y}\Bigg)\Bigg\}\Big(1+\frac{rk\hat{p}_Y}{y_N - r\hat{p}_Y}\Big) - \sum_{k=1}^{K}\Bigg\{\prod_{l=1}^{k}\Bigg(\delta\frac{z_N - z_l}{z_N - r\hat{p}_Z}\Bigg)\Bigg\}\Big(1+\frac{rk\hat{p}_Z}{z_N - r\hat{p}_Z}\Big)\Bigg]. \tag{10}$$

Note that for each $k \leq K$, we have

$$\frac{y_N - y_k}{y_N - r\hat{p}_Y} = \frac{z_N - z_k}{z_N - r\hat{p}_Z},$$
$$\frac{\hat{p}_Y}{y_N - r\hat{p}_Y} > \frac{\hat{p}_Z}{z_N - r\hat{p}_Z}, \tag{11}$$

where (11) holds because $\beta > 0$. Substitute these equations into (10), and we obtain that the derivative is positive. Hence for $p_Y(\eta)$ and $p_Z(\eta)$ with a small $\eta > 0$, we have $\pi^Y + \pi^Z > \pi^T$.

Finally, suppose $(y_K + z_K)/r = p$. In this case, for $p_Y(\eta)$ with a small $\eta > 0$, the RHS of (9) does not represent π^Y (however, it does π^Z). This is because $p_Y(\eta) < y_K/r$ and therefore K must be replaced with $K-1$. However, by (6), this implies that π^Y is *greater* than the value in the RHS of (9). Hence we again have $\pi^Y + \pi^Z > \pi^T$, which proves (iii). Q.E.D.

One particular instance of affinity (but not homogeneity) is tie-in sales of network and nonnetwork goods. In this case, Z is a standard network goods, while Y is a nonnetwork goods with $\alpha = 0$ and $\beta > 0$. β is the value of the nonnetwork goods, which does not depend on the number of adopters. Proposition 5 implies that tie-in sales of those products are inferior to separate sales. Since sales of network goods are subject to delay, the tie-in sales cause part of the value of nonnetwork goods to vanish. This argument, though simple, suggests that tie-in sales are meaningful only when the firm does not establish a strong price control over the nonnetwork goods. Otherwise, the monopolist optimally chooses to sell those goods separately.

4.3 Maxmin Price

Thus far, we have not imposed any theory about, given multiplicity of equilibria in the dynamic coordination game consumers play, which equilibrium will be actually played. Though this agnostic approach does not yield clear-cut predictions about the outcome of the model, we have seen that it still yields some insights on product design of network goods and relevance of tie-in sales. Nevertheless the predictive power of the current model is limited, and if we want to sharpen it, we must introduce some equilibrium selection criterion.

This subsection adopts one extreme selection criterion. That is, we assume that the worst perfect Bayesian equilibrium from the firm's viewpoint is played for any price the firm chooses. The criterion may be interpreted in various manners. It may be that consumers consistently lack ability to coordinate successfully, or it may reflect the firm's pessimistic belief on the consumer behavior.[7] Whatever the interpretation is, now the firm's objective is clear. If we denote the worst equilibrium profit of $\Gamma(p)$ by $\pi(p)$ (as in (4)), the firm chooses a price which maximizes π. We call it the *maxmin price.*

As (4) shows, π exhibits discontinuity at some points, so that existence of a maximizer is not obvious. Indeed, it is easy to construct a set of parameters under which there is no maximum of π.[8] The following result shows that one sufficient condition for existence is patience, or a shorter information lag.

Proposition 6. *Fix network goods* $(z_k)_{k=1}^N$ *and a discount rate* r*. Let* $\hat{N}$ *be a unique integer such that* $z_{\hat{N}-1} < z_{\hat{N}} = z_N$*. Then there exists* $\bar{\tau} > 0$ *such that for any* $\tau \leq \bar{\tau}$*,* $\max_p \pi(p)$ *uniquely exists and it is on the range* $(z_{\hat{N}-1}/r, z_{\hat{N}}/r)$*. Moreover,* $\max_p \pi(p)$ *converges to* $(Nz_N)/r$ *as* $\tau \to 0$*.*

Proof. Fix $(z_k)_{k=1}^N$ and r. Then for any $p < z_{\hat{N}}/r$, we have:

$$\lim_{\tau \to 0} \pi(p) = Np. \tag{12}$$

This is because $\tau \to 0$ implies $\delta \to 1$ for a fixed r, which in turn implies $\exp(-rT_0) \to 1$ in (6).

By (12), for sufficiently small τ, π attains a value strictly greater than $Nz_{\hat{N}-1}/r$ at some p. Since $\pi(p) \leq (Nz_{\hat{N}-1})/r$ for any $p \leq z_{\hat{N}-1}/r$, $\max_p \pi(p)$ is not attained at such a price. Since π is continuous on $(z_{\hat{N}-1}/r, z_{\hat{N}}/r)$, and eventually goes to zero as $p \to z_{\hat{N}}/r$ for a fixed τ, a maximum exists.

On the range $(z_{\hat{N}-1}/r, z_{\hat{N}}/r)$, if $\pi(p) > 0$, it is given by:

$$\pi(p) = Np \cdot \left[1 - \frac{1-\delta}{\delta} \sum_{k=1}^{\hat{N}-1} \left\{ \prod_{l=1}^{k} \left(\delta \frac{z_N - z_l}{z_N - rp} \right) \right\} \right].$$

The first-order condition is:

$$1 - \frac{1-\delta}{\delta} \sum_{k=1}^{\hat{N}-1} \left\{ \prod_{l=1}^{k} \left(\delta \frac{z_N - z_l}{z_N - rp} \right) \right\} \left(1 + \frac{rkp}{z_N - rp} \right) = 0. \tag{13}$$

Since the LHS of (13) is strictly decreasing in p, π is strictly concave on that range. Hence the maximizer is unique.

Finally, (12) implies that any value $v < (Nz_N)/r$ is attained by some $\pi(p)$ if $\tau > 0$ is sufficiently small. Therefore, $\max_p \pi(p)$ converges to $(Nz_N)/r$ as $\tau \to 0$. Q.E.D.

(13) characterizes the maxmin price for all cases where the consumers are sufficiently patient. In general, one cannot solve (13) explicitly. However, we can if $N = 2$,

[7] The latter interpretation applies if the firm is infinitely risk-averse about uncertainty regarding equilibria to be played.

[8] However, a supremum exists. Hence one may want to think that in that case a price approximating the supremum will be selected. Another way to cope with the nonexistence is to assume that the set of prices is finite. Then the optimal price obviously exists, though we cannot characterize it in terms of derivatives.

in which case (13) reduces to:

$$r^2p^2 - 2rz_2p + z_2\{(1-\delta)z_1 + \delta z_2\} = 0. \tag{14}$$

Solving (14) and eliminating a solution greater than z_2/r, we obtain

$$p^* = \frac{z_2 - \sqrt{(1-\delta)z_2(z_2 - z_1)}}{r}, \tag{15}$$

with the maximized profit:

$$\pi^* = \frac{2}{r}\left\{\sqrt{z_2} - \sqrt{(1-\delta)(z_2 - z_1)}\right\}^2. \tag{16}$$

(15) is a solution only if τ is sufficiently small. Some computation proves that it is indeed a solution if:

$$\delta^2 z_2 \geq (2-\delta)^2 z_1. \tag{17}$$

Note that (17) is implicitly a condition on τ, for fixed z_1, z_2 and r. Since it requires δ to be close to 1, it requires τ to be small.

Several properties of the maxmin price easily obtain. First, since the LHS of (13) is strictly increasing in each z_k, so is the maxmin price. The maxmin profit is also increasing in z_k, since we have seen that π uniformly increases (on the range which is positive) with each z_k. Similarly, both maxmin price and profit are decreasing in τ for fixed r. Also (13) shows that the maxmin price is homogeneous of degree 1 with respect to $(z_k)_{k=1}^N$. If each z_k is replaced by αz_k with $\alpha > 0$, then αp solves (13). Thus the multiplied value of network goods simply multiplies the monopolist's price control by the same degree.

In relation to the analysis of Subsection 4.1, one can analyze the firm's investment decision on its qualities, using maxmin profits as values of particular network goods. For example, let us consider investments in stand-alone values (β in (7)). By (13), it is easy to see that when β increases by a certain amount, rp increases by a smaller amount. That is, the firm does not fully extract the value of its increased stand-alone value. This suggests that the firm's investment into stand-alone values tends to be suboptimal from a social viewpoint.

5 Extensions

In this section, we briefly comment on possible extensions of the model. We discuss two assumptions: one is common discount rates, and the other is the assumption that the monopolist commits to its price.

5.1 Heterogeneous Discount Rates

In reality, the firm and the consumers may have differential time preferences. So it would be natural to assume that they have different discount rates; for instance, all consumers have a common discount rate r_C and the firm has r_F. One difficulty of this extension is that Lemma 1 no longer holds, which makes it difficult to characterize the equilibrium profit set. This point is most easily explained if we additionally assume that a public randomization device is available, so that the consumers can correlate their actions. Let r_C be given, and suppose that if $r_F = r_C$, the worst equilibrium profit is attained by a perfect Bayesian equilibrium where all consumers buy at time $T_0 > 0$. Fix a small $\varepsilon > 0$, and consider the following strategy profile. Consumers do not buy until time $T_0 - \varepsilon$, and then depending on the public randomization at time $T_0 - \varepsilon$, they simultaneously buy with probability λ. With the remaining probability, they wait until time $T_0 + \varepsilon$ and then simultaneously buy. λ is chosen so that the expected utility of each consumer equals $\exp(-rT_0)[(z_N/r) - p]$. For small $\varepsilon > 0$, the profile is an equilibrium. If $r_F < r_C$, then the equilibrium gives a smaller profit than the profile that would have been worst if $r_F = r_C$. This is because, in order to keep it indifferent with coordination at time T_0, less patient consumers are content with a smaller probability of earlier coordination. The more patient firm does not like the tradeoff, which results in a smaller profit.

Hence it is much more difficult to characterize the equilibrium profit set under heterogeneous discounting.[9] One resolution is to limit attention to equilibrium with a deterministic path. Then the perfect Bayesian equilibrium with coordination at time T_0 remains to be worst to the firm. Then the same analysis as the one in the text goes through, and results obtained there extend to the case of heterogenous discounting.

5.2 No Commitment

Thus far, our maintained assumption is that the firm commits to its price, and the price is constant over time. This subsection analyzes what happens if the firm cannot commit to a constant price. In this subsection, we adopt the approach in Subsection 4.3, and assume that the firm attempts to choose a price so as to maximize the worst equilibrium profit.

First, we must be careful in specifying what the monopolist can do as its pricing strategy. In dynamic games like this model, it can choose its pricing behavior as a way to reward or punish consumers in order to implement particular outcomes. This aspect makes analysis excessively complicated, and introduces a wide range of behaviors that is well beyond what we want to describe under the assumption of no ability to commit. Hence this subsection assumes that the firm can change its price only at the moment it gets known that some consumer(s) has bought and therefore the number of remaining consumers reduces. That is, if a consumer buys at some time t, then that fact becomes common knowledge at time $t + \tau$, at which the firm may change its price. The new price becomes effective from time $t + \tau$ on (inclusive), until some other consumer's purchase is recognized.

If the firm can change its price in that way, a typical phenomenon is that it *raises* the price after additional purchase is made, because the purchase makes the network goods more valuable to remaining consumers.[10] In particular, it wants to charge a

[9] We do not know if similar examples are constructed without a public randomization.

[10] This is in contrast with the Coase conjecture (Gul, Sonnenschein, and Wilson, 1986, for example). In our model, where the consumers are homogenous and network effects are present, their (average) willingness to pay after some purchases increases, while in the Coase conjecture context it decreases

price that is optimal in a model with the remaining number of consumers and with the corresponding valuation profile (assuming that no commitment is possible in the continuation game). Hence we solve the whole game backward, but there is one subtle issue; how to deal with subgames with exactly one consumer remaining? Here we suppose that the firm charges the full value of the network goods to the consumer and he immediately buys it. This seems to be a contradiction against our assumption of pessimistic seller, because at that price the consumer is indifferent between buying it and not doing so. Still we adopt that supposition in order to avoid nonexistence. One explanation is that the monopolist's pessimism is completely ascribed to the coordination problem on the side of consumers.

Under the above assumption for the one-consumer subgames, it immediately follows that if there are only two consumers, then whether the firm can commit to its price or not has no effect on the maxmin price (if any). The reason is that possibility of commitment does not change the value a consumer can guarantee in any perfect Bayesian equilibrium of the dynamic coordination game (namely, the value in (2)). If a consumer buys immediately, then at time τ the firm adjusts its price to z_2/r, and the other consumer buys immediately at the new price, by the above supposition. Hence the consumers' response to early purchase does not depend on whether commitment is possible or not, and the same argument as Proposition 1 goes through. Thus for lack of commitment to matter, we need three or more consumers, so hereafter we consider the case $N = 3$ as the simplest one.

If $N = 3$ and the firm cannot commit to its price, then the payoff level a consumer can guarantee in any perfect Bayesian equilibrium is modified as follows. If a consumer buys at time 0 and that is recognized at time τ, then the firm changes its price so that the worst perfect Bayesian equilibrium under the new price is such that the two consumers buy at time $S + \tau$, where S satisfies

$$e^{-rS} = 1 - \sqrt{(1-\delta)(1-\frac{z_2}{z_3})}. \tag{18}$$

(18) follows from (2) and (15). Hence if a consumer buys at time 0, she can guarantee the payoff of:

$$\frac{(1-\delta e^{-rS})(z_1 - rp) + \delta e^{-rS}(z_3 - rp)}{r}.$$

Proceeding similarly, we can show that the worst equilibrium profit is attained by a profile where the consumers simultaneously buy at time T_0 such that:

$$\frac{(1-\delta e^{-rS})z_1 + \delta e^{-rS} z_3}{r} - p = e^{-rT_0}\left(\frac{z_3}{r} - p\right). \tag{19}$$

Using (18) and (19), we can compute the firm's profit:

$$3p \cdot \frac{z_1 - rp + \delta(z_3 - z_1)\left\{1 - \sqrt{(1-\delta)(1-\frac{z_2}{z_3})}\right\}}{z_3 - rp}. \tag{20}$$

due to exit of high-type consumers.

We compare (20) with the profit function π used in the previous analysis, using some numerical examples. Suppose $z_k = k^2$ for each k, $r = 0.01$ and $\delta = 0.99$. It is easy to verify that (20) attains a maximum at $p = 654.4$, with the maximized profit of 475.8. If the monopolist can commit to a price under the same parameters, then the optimal price is a solution of (13), and is computed as $p = 701.7$. The corresponding profit is 602.7. This shows that the monopolist suffers from lack of commitment, which must charge a lower price only for a smaller profit.

6 Conclusion

We have developed a theory of monopoly pricing of durable network goods. Our supposition was that the monopolist takes into account a possibility of coordination failure on the side of consumers. We first adopted a position to assume that the firm has no particular theory as to which equilibrium will be played. We characterized all possible equilibrium outcomes in terms of the firm's profit, and applied the result to issues such as relevance of tie-in sales. Then we shifted to a more extreme position to assume that the consumers commit dynamic coordination failure in the worst way to the firm. While this type of equilibrium behavior has received little attention in the literature, its introduction allows us to derive an optimal price and has rich implications on the role of stand-alone values, information lag and patience, etc.

A popular approach to dynamic coordination problems is to introduce incomplete information (Cabral, Salant, and Woroch, 1999; Ochs and Park, 2004; Xue, 2003, for example), partly because under incomplete information the "standard" type of equilibrium, which would correspond to the maximum profit in our model, yields rich implications. However, the dynamic coordination problems and multiplicity of equilibria remain even in those frameworks, and it is typical in the literature to (often implicitly) exclude other equilibria. Furthermore, to the extent that implications of the popular approach depend on the way incomplete information is modeled, the empirically testable predictions are hard to obtain. Our point is that even apparently too simple models of complete information can produce a meaningful theory on the network goods pricing.

A Appendix: Proof of Lemma 2

Fix a perfect Bayesian equilibrium of $\Gamma(p)$ and a history at time t arbitrarily. The proof is by induction on the number of other consumers who are known to have bought at the history, denoted by $k = 0, 1, \cdots, N-1$.

If $k = N-1$, we only need to consider the case where $p < z_N/r$. In this case, the unique optimal continuation strategy is to buy immediately. Since $T_{N-1} = 0$, the claim follows. Next, fix $k \geq K$ and suppose that the claim holds for any $k' \geq k+1$. Then by the same argument which proves Proposition 1(i), we can show that the unique sequentially rational continuation strategy is to buy immediately. Since this applies to any remaining consumer and since $T_k = 0$, the claim again follows.

Now suppose that the claim holds for some $k \leq K$ and any $k' \geq k$, and the consumer is at a history where $k-1$ consumers have already bought. Suppose she buys immediately. Given that, let $p_\kappa(t')$ ($t' \geq t$) be the probability that exactly κ consumers have bought by time t'. Note that $p_\kappa(t') = 0$ for any $\kappa < k$ and any $t' \geq t$. Also let $p_{k'}(t';\kappa)$, where $t' \geq t+\tau$ and $k' \geq \kappa$, be the probability that exactly k' consumers have bought by time t', conditional on that exactly κ consumers have bought by time $t+\tau$.

Consider time $t+\tau$. Then with probability $p_\kappa(t+\tau)$ ($\kappa \geq k$), there are exactly $N-\kappa$ consumers who have yet to buy, knowing that k or more consumers have already bought. Note that T_k is decreasing in k. Thus by the induction hypothesis, each of $N-\kappa$ consumers is guaranteed the continuation payoff of $\exp(-rT_k)[(z_N/r)-p]$. Hence we have:

$$(N-\kappa)e^{-rT_k}\left(\frac{z_N}{r}-p\right) \leq \int_{t+\tau}^{\infty}\Big\{\sum_{k'=\kappa}^{N} p_{k'}(t';\kappa)(k'-\kappa)(z_{k'}-rp)\Big\}e^{-r(t'-t-\tau)}dt' \quad (21)$$

$$\leq \int_{t+\tau}^{\infty}\Big\{\sum_{k'=\kappa}^{N} p_{k'}(t';\kappa)(k'-\kappa)(z_N-rp)\Big\}e^{-r(t'-t-\tau)}dt', \quad (22)$$

where the second inequality follows from monotonicity of z_k's. Since (22) implies

$$(N-\kappa)\frac{e^{-rT_k}}{r} \leq \int_{t+\tau}^{\infty}\Big\{\sum_{k'=\kappa}^{N} p_{k'}(t';\kappa)(k'-\kappa)\Big\}e^{-r(t'-t-\tau)}dt',$$

we invoke monotonicity of z_k's and manipulate (21) as follows.

$$\frac{(1-e^{-rT_k})(z_k-rp)+e^{-rT_k}(z_N-rp)}{r} \leq \int_{t+\tau}^{\infty}\Big\{\sum_{k'=\kappa}^{N} p_{k'}(t';\kappa)(z_{k'}-rp)\Big\}e^{-r(t'-t-\tau)}dt'$$

$$= \int_{t+\tau}^{\infty}\Big\{\sum_{k'=k}^{N} p_{k'}(t')(z_{k'}-rp)\Big\}e^{-r(t'-t-\tau)}dt'. \quad (23)$$

Here the second equality follows from taking expectation of the first inequality with respect to $p_\kappa(t+\tau)$.

Using (23), we can evaluate of the payoff of a consumer who buys at time t as:

$$\int_{t}^{\infty}\Big\{\sum_{k'=k}^{N} p_{k'}(t')(z_{k'}-rp)\Big\}e^{-r(t'-t)}dt'$$

$$\geq \int_{t}^{t+\tau}(z_k-rp)e^{-r(t'-t)}dt' + \delta\int_{t+\tau}^{\infty}\Big\{\sum_{k'=k}^{N} p_{k'}(t')(z_{k'}-rp)\Big\}e^{-r(t'-t-\tau)}dt'$$

$$\geq \frac{(1-\delta e^{-rT_k})(z_k-rp)+\delta e^{-rT_k}(z_N-rp)}{r} = e^{-rT_{k-1}}\left(\frac{z_N}{r}-p\right),$$

where the last equality is by (5). Since her equilibrium continuation payoff is no less than this, the claim holds.

B Appendix: Other Types of Equilibria in $\Gamma(p)$

We first provide an example of a mixed strategy perfect Bayesian equilibrium. Let $N = 2$, and let p be:

$$\frac{z_1}{r} < p < \frac{(1-\delta)z_1 + \delta z_2}{r}.$$

Each consumer plays the following strategy. At time 0, she buys with probability λ such that:

$$\lambda = \frac{rp - z_1}{z_2 - z_1} \in (0, 1).$$

At time $t \in (0, \tau)$, she does not buy. And at time $t \geq \tau$, she buys.

The pair of these strategies is a perfect Bayesian equilibrium. At time $t \geq \tau$, sequential rationality is trivial. At time $t \in [0, \tau)$, the only consistent belief about the other consumer's continuation play is that (i) by time τ, he buys with probability λ, and (ii) from time τ on (inclusive), he buys with probability 1. Since $t < \tau$, a consumer's continuation play does not affect the other consumer's play. Since instantaneous payoffs on the length $[t, \tau)$ given the belief is zero by the definition of λ, she is indifferent between buying now or not. Hence her continuation strategy is optimal, and therefore the profile is a perfect Bayesian equilibrium.

We next give an example of an asymmetric equilibrium.[11] Let $N = 3$, and assume

$$\frac{z_2}{r} < p < \frac{(1-\delta)z_1 + \delta z_3}{r}, \quad 0 < T_0 < \infty, \tag{24}$$

where T_0 is given by (6). For a fixed (z_1, z_2, z_3) with $z_2 < z_3$, we can choose p satisfying (24) if δ is sufficiently close to 1.

Let $\hat{\sigma}$ be the perfect Bayesian equilibrium of $\Gamma(p)$ specified in the proof of Proposition 3, where all consumers simultaneously buy at time T_0. We modify $\hat{\sigma}$ as follows: (i) consumer 1 buys at time 0, and (ii) at time τ, both consumers 2 and 3 buy if they know that consumer 1 has bought. At any other history, the strategies specify the same actions as $\hat{\sigma}$. Note that the new profile has the path that consumer 1 buys at time 0 and then the other consumers simultaneously buy at time τ. Also note that under the profile, if consumer 1 deviates and does not buy until time τ, then the continuation play is such that all consumers simultaneously buy at time T_0.

We show this profile is a perfect Bayesian equilibrium. First, at any history at time $t > \tau$, all consumers' continuation strategies are the same as $\hat{\sigma}$. Hence they are clearly sequentially rational. Second, consider a history at time τ. Then for consumer 2 or 3, if she knows that consumer 1 has bought, then it is optimal to buy now because the other consumer will also buy now. If she does not observe consumer 1's purchase, then all consumers' continuation strategies are the same as $\hat{\sigma}$, sequential rationality follows. Third, consider a history at time $t \in [0, \tau)$. For consumer $j \geq 2$, the only consistent belief of the consumer is that consumer 1 has already bought and the remaining consumer will buy at time τ. Hence the best reply is to buy at time τ, so not buying now (as is specified by $\hat{\sigma}$) is optimal.

Finally, consider consumer 1. If $t \neq 0$, then all consumers' continuation strategies

[11] We conjecture that all perfect Bayesian equilibria are symmetric if $N = 2$.

are the same as $\hat{\sigma}$, her continuation play is sequentially rational. If $t = 0$, suppose consumer 1 does not buy now and then conforms to the strategy. This gives the payoff of $\hat{\sigma}$, which can be computed as:

$$\frac{(z_1 - rp)(z_3 - rp) + \delta(z_3 - z_1)\big[(1-\delta)z_2 + \delta z_3 - rp\big]}{r(z_3 - rp)} < \frac{z_1 - rp + \delta(z_3 - z_1)}{r}, \quad (25)$$

where the RHS of (25) is her payoff of buying now. Hence buying now is sequentially rational, and the profile is a perfect Bayesian equilibrium.

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