On the axiomatic distribution of opportunities *

Jorge Alcalde-Unzu[1]　　Miguel A. Ballester[2]

August 2006

[1] Departamento de Economía, Universidad Pública de Navarra. Campus de Arrosadia, Pamplona, 31006, Spain. E-mail: jorge.alcalde@unavarra.es

[2] Corresponding author. Departamento d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona. Edifici B 08193 Bellaterra (Barcelona). Spain E-mail: mballester@pareto.uab.es

Running title: On ranking opportunity distributions

*We are very grateful to John Weymark, Laurence Kranich, Jorge Nieto and Ricardo Arlegi for their very helpful comments and suggestions. Financial support from the Spanish Ministry of Education through grants SEC2003-08105, SEJ2005-01481/ECON, FEDER and the Barcelona Economics Program of CREA is gratefully acknowledged.

Abstract

This paper considers the goodness of profiles of opportunity sets taking into account some relevant features of this distributive problem in addition to equity. Apart from some basic axioms, we consider additional properties that make sense when the unique information available to the social decision-maker to rank the opportunity sets is the partial inclusion ranking. We show that the unique way to evaluate the distribution of opportunities in that case is to sum the number of opportunities available to all individuals in the society. We show two ways to overcome this result. In each one, we characterize two families of criteria such that they include many of the apparently disparate rankings proposed in the literature.

1 Introduction

Individuals in any society require opportunities to fulfill their needs and desires. The availability of opportunities is probably one of the basic requirements to make life worthwhile. Thus, the distribution of opportunities is a significant feature of a social choice problem. In view of the wide range of interpretations of the notion of opportunity (from Rawls' primary goods to Sen's capabilities), the general framework adopted in our study is one that can be adapted to all of them. Individuals are endowed with opportunity sets, and social situations are evaluated in terms of their opportunity profiles, i.e., the description of the opportunities available to each individual.

Despite the simple structure of this model, the literature proposes several social rankings to evaluate the distribution of opportunities (for a survey, see Peragine [9]). All of these proposals identify some of the relevant features of the social problem that may influence the social decision-maker. Most of these rankings highlight the importance of equality of opportunities as a key issue in the social evaluation of public policies. This is the basic point in the proposals of Kranich [5] and Herrero [3], where some characterizations of differences-based rules are provided. The proposals of Herrero et al. [4] and Bossert et al. [1] also consider equity issues through their interest in the opportunities common to all individuals. Equality is also the main concern in Ok [6], Ok and Kranich [7] and Savaglio and Vanucci [11], where the aim is to describe the parallelism between the opportunity framework and the more classical income distribution approach. While Weymark [13] also exploits this parallelism, he is concerned with providing an overall evaluation of opportunity distributions. Such an evaluation would cover aspects of the distribution of opportunities in addition to equity, such as the availability of opportunities (an *efficiency* aspect). We adopt this more general view in our paper.

Furthermore, all of this literature shares two important features that we will attempt to highlight in the course of this paper. First, despite the apparent differences between all of the proposed rankings, we will show throughout the paper that there is a core of properties common to all of them. This fact will be reflected in this paper by a group of axioms capturing the intuitive ideas about the goodness of opportunity distributions underlying many of the axioms in the literature. We will propose weaker versions of some of these axioms in order to achieve the above-mentioned aim of avoiding considering equity as the only relevant aspect of goodness.

Second, societies are ranked in the proposals of the literature on the basis of absolute criteria; i.e., the implementation of a public policy that involves a switch from one distribution to another is judged independently of the status quo position. We incorporate this idea to our analysis through some properties that will be introduced in the next sections.

Additionally, in the evaluation of opportunity distributions, it is important to have a ranking to evaluate the opportunity sets of the agents in order to know which individuals are advantaged and which are disadvantaged in the distribution. Ok [6] showed that the unique *complete* ranking of opportunity sets that is compatible with some properties of equality in opportunity profiles is the Cardinality-based Criterion of Pattanaik and Xu [8]. Given this result and the fact that the unique unquestionable comparisons for the social decision-maker are the ones implied by the partial inclusion ranking, we propose as a starting point this *partial* ranking to judge the desirability of the opportunity sets.

Then, the objective of this paper is to find general absolute rankings of opportunity distributions using the partial inclusion ranking to evaluate the opportunity sets. We show that these assumptions lead to the sum of the number of opportunities of all individuals as the measure of the goodness of an opportunity distribution. Given that we think that this ranking is not very concerned with equity issues, we propose two ways to overcome this result. First, we propose that the criterion incorporates some information about the common alternatives of the distribution when it judges a public policy. This consideration leads to a family of intuitive criteria that includes some of the proposals of the literature. Second, we propose to consider a complete ranking of opportunity sets as a starting point. Given the results of Ok [6], we select the Cardinality-Based Criterion and we obtain other family of intuitive criteria, including other proposals of the literature.

The structure of the paper is as follows. Section 2 presents some notation and definitions. Section 3 is devoted to presenting the two-agent case. After that, in Section 4, all of the axioms are intuitively generalized to the case of an arbitrary number of agents. Some extensions of the results for the two-agent case to this more general framework are provided. Section 5 offers a brief conclusion. The proofs of the results are presented in the Appendix.

2 Basic Notation and Definitions

Let $I = \{1, \ldots, n\}$ denote the finite set of individuals and let X be an infinite set of opportunities. The set of non-empty finite subsets of X is L. An opportunity set for agent $i \in I$ is an element $O_i \in L$. We consider profiles of opportunity sets of the form $O = (O_1, \ldots, O_n) \in L^n$. Let $O^{\cup} = \bigcup_{i \in I} O_i$ and $O^{\cap} = \bigcap_{i \in I} O_i$.

Given $O \in L^n$, σ^O denotes the set of permutations σ of I such that $|O_{\sigma(i+1)}| \leq |O_{\sigma(i)}|$ for all $i \in \{1, \ldots, n-1\}$. The profile $(O_{\sigma(1)}, \ldots, O_{\sigma(n)})$ is denoted by $\sigma(O)$. A profile O is *nested* if $O_{\sigma(i+1)} \subseteq O_{\sigma(i)}$ for all $i \in \{1, \ldots, n-1\}$ and for all $\sigma \in \sigma^O$.[1] In particular, we denote by $\mathcal{N}$ the set of nested profiles in which the identity mapping belongs to σ^O. That is, $\mathcal{N} = \{O \in L^n \text{ such that } O_{i+1} \subseteq O_i \text{ for all } i \in \{1, \ldots, n-1\}\}$.

Along the paper, we consider changes involving the addition and removal of opportunities for profiles in $\mathcal{N}$. To simplify the exposition for some of

[1] Notice that this is equivalent to require inclusion only for some $\sigma \in \sigma^O$. This equivalence allows us, for the sake of simplicity, to avoid any quantifier involving σ in the formulation of our properties.

these cases, we define a *change function* as a map $f : (L \cup \emptyset)^{2n} \times \mathcal{N} \to (L \cup \emptyset)^n$ such that for all $A_1, B_1, \ldots, A_n, B_n \in (L \cup \emptyset)$ and for all $O \in \mathcal{N}$, $f(A_1, B_1, \ldots, A_n, B_n, O) = ((O_1 \setminus A_1) \cup B_1, \ldots, (O_n \setminus A_n) \cup B_n)$.

When clear, we do not explicit the added and removed elements and use $f(\cdot, O)$ as notation. We also denote the i-th component of $f(\cdot, O)$ as $f_i(\cdot, O)$.

We consider transitive and complete binary relations $\succsim \subseteq L^n \times L^n$. $O \succsim U$ is used to capture the general idea that profile O is socially preferred to profile U. The relations $\succ$ and $\sim$ are defined as usual.

The set of natural numbers is denoted by $\mathbb{N}$, whereas $\mathbb{N}^* = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rational numbers and $\mathbb{R}$ is the set of real numbers.

3 The two-agent framework

We initially focus exclusively on the case of two agents, $n = 2$. The following three axioms are weaker versions of various axioms found in the literature.

Anonymity (ANON): $(O_1, O_2) \sim (O_2, O_1)$ for all $O \in L^2$.

Anonymity is a very basic property when distributional concerns are important. There is no proposal in this literature that denies the relevance of the Anonymity axiom. In the acceptance of ANON is implicit that all the relevant characteristics to judge social situations are included in the opportunity sets of the individuals.

Assimilation (ASM): For all $O \in L^2$, all $x \in (X \setminus O^{\cup})$ and all $y_1, y_2 \in X$, we have that $(O_1 \cup \{x\}, O_2 \cup \{x\}) \succsim (O_1 \cup \{y_1\}, O_2 \cup \{y_2\})$ whenever:

1. $y_i \in O_i$ for all $i \in \{1, 2\}$

or

2. $y_i \notin O_i$ for all $i \in \{1, 2\}$ and there exists $j \in \{1, 2\}$ such that $y_j \notin O^{\cup}$.

Assimilation is a relaxed version of a combination of two different properties (Independence of Common Expansions and Assimilation) proposed in the original work of Kranich [5] and also incorporated in Weymark [13]. Independence of Common Expansions implies that the social evaluation ordering remains invariant after the addition of the same element to both individuals. Case 1 in our axiom is a weaker form of this requirement because it allows a social improvement in such situations. A variety of reasons can be used to support this view. For instance, we could appeal to a concern for other ethical values apart from equity (efficiency, for instance). It can also be argued, even in a case concerning pure equity, that equal expansions lead to reductions in inequality, considered as a *relative* notion.

Case 2 in our property implies that the addition of any two new opportunities (one for each individual) in a given profile is not strictly better than offering the same new opportunity to both individuals. This is a weaker axiom than the Assimilation axiom in Kranich [5] and in Weymark [13]. The axiom they propose implies the following:

For all $O \in L^2$ and all $x, y_1, y_2 \in X$ such that $y_1 \notin O_1$, $y_2 \notin O_2$ and $x \notin O^{\cup}$,

$$(O_1 \cup \{x\}, O_2 \cup \{x\}) \succsim (O_1 \cup \{y_1\}, O_2 \cup \{y_2\}).$$

Then, we have that case 2 of our property is weaker than the axiom in Kranich [5] and in Weymark [13], because they require to the alternatives y_1 and y_2 only that $y_i \notin O_i$ for all $i \in \{1,2\}$, whereas we impose the additional requirement that there must be $j \in \{1,2\}$ such that $y_j \notin O^{\cup}$. This weakening is not trivial, since it avoids some counterintuitive results that may be implied by Kranich's and Weymark's axiom. For example, consider the profile $O = (\{y_2\}, \{y_1\})$ and the following expansions of O; first, add the alternative y_1 to the first individual and y_2 to the second one, obtaining the profile $O' = (\{y_1, y_2\}, \{y_1, y_2\})$. Second, consider the expansion of O implied by the addition of x to both individuals. The resulting profile is

$O'' = (\{y_2, x\}, \{y_1, x\})$. If the spirit of the axiom is giving more importance to the common alternatives in the evaluation of the equality of opportunities, O' should be better than O''. The reason is that O' is a situation in which both individuals have the same opportunity set of two alternatives and in O'' both individuals have two opportunities, but only one is common. However, the Assimilation axiom of Kranich [5] and Weymark [13] states that $O'' \succsim O'$, while our case 2 does not support any preference between O' and O'', given that $y_1 \in O_2$ and $y_2 \in O_1$.

We think that our version avoids these paradoxes because it requires that at least one of the alternatives y_1 and y_2 must be totally exogenous to the profile O, whereas the version of Kranich [5] and Weymark [13] allows that $y_1 \in O_2$ and $y_2 \in O_1$.

Monotonicity (MON): For all $O \in \mathcal{N}$ and for all $x \notin O_2$,

$$(O_1, O_2 \cup \{x\}) \succsim (O_1 \cup \{x\}, O_2), \text{ with strict preference if } x \in O_1.$$

Monotonicity expresses a social concern for disadvantaged individuals. Suppose that the opportunity set of agent 2 is strictly contained in that of agent 1. Then, in any arguable sense, we can say that individual 1 is advantaged with respect to individual 2. The axiom says that, in such situations, it is weakly preferable to offer a new opportunity to the disadvantaged individual than to the advantaged one. With the same objective, the reduction of an existing gap between agents 1 and 2 (by offering an alternative in $O_1 \setminus O_2$ to the disadvantaged agent) should be considered a social improvement. The first part of the axiom resembles the Generalized Pigou-Dalton Transfer Principle of Weymark [13]. With respect to the second part, the reduction of existing gaps is also the motivation of Kranich's Monotonicity axiom. However, in Kranich [5], it is proposed that the addition of an alternative to the advantaged agent constitutes a strict reduction in total social welfare. We consider our version to be more appropriate within a general approach in which equity may not be the only social evaluation criterion.

These three axioms are satisfied by a large collection of rankings and will be the core of properties that share some of the rankings of the literature. We now propose two more properties to be incorporated into the analysis. They are not as unquestionable as ANON, ASM or MON, but may make sense in a context in which the social decision-maker has, as the unique information to rank opportunity sets, the partial inclusion ranking.

Absence of Information in Non-Nested Profiles (AIN): For all $O \in L^2$ such that $\sigma(O) \notin \mathcal{N}$ and for all $x_i \notin O_i$, with $i \in \{1,2\}$, such that $x_1 \in O_2 \Leftrightarrow x_2 \in O_1$,

$$(O_1 \cup \{x_1\}, O_2) \sim (O_1, O_2 \cup \{x_2\}).$$

As we have mentioned above, AIN makes sense only whenever the inclusion partial order is the only relevant (social) information to rank opportunity sets. In this case, we have no information about which individual is more disadvantaged except in nested profiles. AIN implies that, in the absence of such information, the evaluation of an enhancement to an individual opportunity set should be ranked independently of the agent involved. We require to such enhancements that if the opportunity added to the first individual belongs to the opportunity set of the second individual, then the alternative added to the second individual must belong to the opportunity set of the first one, and vice versa. The justification of the property comes from the fact that, in non-nested profiles, the social decision-maker is not able to distinguish between the addition of an alternative to one individual or to another, because she is totally uncertain about who is advantaged and who is disadvantaged.

The following property requires that the social evaluation remains invariant to the application of a common change to two nested profiles.

Independence of Status Quo (ISQ): For all $O, U \in \mathcal{N}$ and for all $A_i, B_i \in (L \cup \emptyset)$ such that $A_i \subseteq (O_i \cap U_i)$, $B_i \cap (O_i \cup U_i) = \emptyset$ for all $i \in \{1,2\}$ and $f(\cdot, O), f(\cdot, U) \in L^2$ whenever

$$f_1(\cdot,O) \not\subset f_2(\cdot,O) \text{ and } f_1(\cdot,U) \not\subset f_2(\cdot,U),$$

it must follow that $O \succsim U \Rightarrow f(\cdot,O) \succsim f(\cdot,U)$.

As we have mentioned in the Introduction, we are interested in this paper in absolute rankings, that is, in criteria that judge the changes between opportunity profiles independently of the status quo position. To reflect this idea, ISQ states that if we apply the same change to two nested profiles, the resultant profiles must be compared in the same way as the original ones. Note that the change must be applied to nested profiles because this is the unique situation in which, given our assumptions, we can differentiate between the advantaged and the disadvantaged agent. This is also the case of the Independence of Rank-Preserving Expansions' property of Kranich [5] and Weymark [13], although they only require the property to hold whenever $A_2 \subseteq A_1$ and $B_2 \subseteq B_1$. Given our focus on the partial inclusion ranking, we expand this idea to all cases in which the original notion of advantaged-disadvantaged does not totally reverse; that is, to situations in which the disadvantaged individual has not converted into the advantaged one (its opportunity set after the change should not be a superset of the opportunity set of the other individual).

An unexpected result

We have described a set of properties to rank opportunity distributions. ANON, ASM and MON reflect fundamental ideas of the literature in a weak version. Our objective is to know which are the consequences of assuming also AIN and/or ISQ, that might make sense only in the context of absolute criteria and when the unique information available to the social decision-maker to rank opportunity sets is the partial inclusion ranking. To do that, consider the following additional characteristic of a ranking of opportunity profiles.

Definition 3.1 *A ranking $\succsim$ on L^2 satisfies the Sum of Opportunities property when for all $O, U \in L^2$, $|O_1| + |O_2| > |U_1| + |U_2| \Rightarrow O \succ U$.*

As will be proved, any criterion satisfying the axioms introduced above also satisfies the Sum of Opportunities property. This seems to be a somewhat deceptive result because this property leads us directly to a compromise that has little to do with equity issues. One is tempted to think that the AIN axiom is the main source of this result. This idea is false, however, because the other four axioms also lead to the same result.

Theorem 3.1 *If $\succsim$ satisfies ANON, ASM, MON and ISQ, then it also satisfies the Sum of Opportunities property.*

Then, given the result of Theorem 3.1, if we desire general criteria for ranking opportunity profiles with different compromises between equity and efficiency, we have to weaken some of the axioms. Given that ANON, ASM and MON are weaker versions of properties documented in the literature, we focus on ISQ.

A possibility result

The ISQ axiom allows changes in which the common opportunity sets ($O^{\cap}$ and $U^{\cap}$) are modified in a substantively different way. As it is widely acknowledged in the literature, the set of common opportunities plays a substantial role when judging social distributions. (See for instance Herrero et al. [4] and Bossert et al. [1], among others.) We may think in some restrictions for the set of appliable changes and we propose the following first weakening of ISQ.

Weak Independence of Status Quo (WISQ): For all $O, U \in \mathcal{N}$ and for all $A_i, B_i \in (L \cup \emptyset)$ such that $A_i \subseteq (O_i \cap U_i)$, $B_i \cap (O_i \cup U_i) = \emptyset$ for all $i \in \{1, 2\}$ and $f(\cdot, O), f(\cdot, U) \in L^2$ whenever

$$f_1(\cdot, O) \not\subset f_2(\cdot, O) \text{ and } f_1(\cdot, U) \not\subset f_2(\cdot, U)$$

and

$$O^{\cap} \setminus [f(\cdot, O)]^{\cap} = U^{\cap} \setminus [f(\cdot, U)]^{\cap} \text{ and } [f(\cdot, O)]^{\cap} \setminus O^{\cap} = [f(\cdot, U)]^{\cap} \setminus U^{\cap},$$

it must follow that $O \succsim U \Rightarrow f(\cdot, O) \succsim f(\cdot, U)$.

We require, as in ISQ, that the notion of advantaged-disadvantaged does not totally reverse acording to the partial inclusion ranking. But now, the opportunities that have disappeared from the common set must be the same in both profiles and the same must occur with the new opportunities incorporated to the common set. Only in this case, WISQ ranks the resulting profiles in the same manner that the original profiles were ranked.

Substituting WISQ for ISQ allows us to escape from the result of Theorem 3.1. Consider the following family of criteria that combine, using a parameter γ, the number of common opportunities and the number of total opportunities in the distribution.

Definition 3.2 *A ranking $\succsim_\gamma$ on L^2 is a* Common Welfare Criterion *with parameter $\gamma \in [-1, 1]$ if and only if it can be expressed in the following way: for all $O, U \in L^2$,*

$$O \succsim_\gamma U \Leftrightarrow |O^\cap| + \gamma \cdot |O^\cup| \geq |U^\cap| + \gamma \cdot |U^\cup|.$$

The Common Welfare Criteria evaluate an opportunity distribution by means of a weighted sum of the number of opportunities in the common set and the number of opportunities in the union set. We also propose other families representing lexicographic refinements of the Common Welfare Criteria, using the extreme cases as tie-breakers whenever the Common Welfare Criterion of a given parameter γ ranks two profiles as indifferent.

Definition 3.3 *A ranking $\succsim^1_\gamma$ on L^2 is a* type 1 Common-Lexicographic Welfare Criterion *with parameter $\gamma \in [-1, 1]$ if and only if it can be expressed as follows: for all $O, U \in L^2$,*

$$O \succsim^1_\gamma U \Leftrightarrow O \succ_\gamma U \text{ or } [O \sim_\gamma U \text{ and } O \succsim_{\gamma=-1} U].$$

Definition 3.4 *A ranking* $\succsim^2_\gamma$ *on* L^2 *is a* type 2 Common-Lexicographic Welfare Criterion *with parameter* $\gamma \in [-1, 1]$ *if and only if it can be expressed as follows: for all* $O, U \in L^2$,

$$O \succsim^2_\gamma U \Leftrightarrow O \succ_\gamma U \text{ or } [O \sim_\gamma U \text{ and } O \succsim_{\gamma=1} U].$$

Only whenever the previously defined weighted sum give the same value for two profiles, $\succsim_\gamma$, $\succsim^1_\gamma$ and $\succsim^2_\gamma$ differ.[2] $\succsim_\gamma$ ranks them as indifferent, meanwhile the other two look for more information in the profiles to establish a preference. To do that, they take into consideration a different weighted sum to break the tie. In case of $\succsim^1_\gamma$, the tie-breaker is $\succsim_{\gamma=-1}$, whereas in case of $\succsim^2_\gamma$, the tie-breaker is $\succsim_{\gamma=1}$. It could be argued that any other non-extreme criterion of the family could be used as a tie-breaker. It can be easily proved, however, that this would be equivalent to using one of the extremes as a tie-breaker, and would in any case lead to the type 1 or 2 Common-Lexicographic Welfare Criterion.

Some relevant rankings are included in these families. The first is the ranking that evaluates the opportunity profiles on the basis of the symmetric difference, i.e., the opportunities available to some but not all individuals in the society. These alternatives reflect somehow the discrimination of agents in society. Counting negatively these alternatives would correspond to the Common Welfare Criterion with a parameter value $\gamma = -1$. This proposal appears to focus exclusively on the equity component, but there are other proposals that include some efficiency aspects. For example, the ranking that evaluates opportunity distributions by the intersection set (or common set) of each distribution is the main point in the article by Herrero et al. [4] and of a criterion proposed by Bossert et al. [1]. This proposal corresponds to

[2]Clearly, this analysis holds for all $\gamma \in \mathbb{Q} \cap (-1, 1)$. If $\gamma \in \{-1, 1\}$, one of the refinements coincides with the original criterion. If $\gamma \in (\mathbb{R} \setminus \mathbb{Q}) \cap (-1, 1)$, both refinements coincide with the original criterion.

the Common Welfare Criterion with a parameter value of $\gamma = 0$.[3] A straightforward manipulation shows also that the Common Welfare Criterion with $\gamma = 1$ judges social distributions in terms of the total number of opportunities available to the individuals (with the common opportunities being counted separately for each individual). One may interpret the parameter γ as the degree of importance given to efficiency issues in the social evaluation.

Herrero et al. [4] also consider the lexicographic evaluation of the common set and the union set in the *utilitarian opportunity relation*, which corresponds to the type 2 Common-Lexicographic Welfare Criterion with $\gamma = 0$.[4]

Indeed, in combination with AIN, axioms ANON, ASM, MON and WISQ characterize the introduced families.

Theorem 3.2 $\succsim$ *satisfies ANON, ASM, MON, AIN and WISQ if and only if there exists* $\gamma \in [-1, 1]$ *such that* $\succsim \in \{\succsim_\gamma, \succsim^1_\gamma, \succsim^2_\gamma\}$.

Another possibility result

In this section, we explore another plausible modification of ISQ. The original property required that the changes did not turn the advantaged agent into the disadavantaged one in any of the profiles (with respect to the partial inclusion ranking). However, the agents involved may have changed their positions with respect to some completion of the partial inclusion ordering. We propose to weaken ISQ to apply only to changes in which the stricter notion of advantaged-disadvantaged reflected by means of the Cardinality-based Criterion is respected. The reasons to choose this particular completion of the partial inclusion ranking are the results in Ok [6]. There, it is proved that this criterion is the unique complete ranking of opportunity sets compatible with some equity properties.

[3]These papers consider this evaluation in terms of a general ranking over opportunity sets, but the criterion corresponds with the Common Welfare Criterion with $\gamma = 0$ whenever the Cardinality-based criterion of Pattanaik and Xu [8] is at stake.

[4]The previous footnote does also apply to this criterion in a parallel way.

Weak Independence of Status Quo 2 (WISQ2): For all $O, U \in \mathcal{N}$ and for all $A_i, B_i \in (L \cup \emptyset)$ such that $A_i \subseteq (O_i \cap U_i)$, $B_i \cap (O_i \cup U_i) = \emptyset$ for all $i \in \{1, 2\}$ and $f(\cdot, O), f(\cdot, U) \in L^2$ whenever

$$|f_1(\cdot, O)| \geq |f_2(\cdot, O)| \text{ and } |f_1(\cdot, U)| \geq |f_2(\cdot, U)|,$$

it must follow that $O \succsim U \Rightarrow f(\cdot, O) \succsim f(\cdot, U)$.

We require, as in axiom ISQ, that the partial notion of advantaged-disadvantaged does not totally reverse acording to the partial inclusion ranking. But now, moreover, WISQ2 strengths this requirement by imposing that the disadvantaged agent does not become the agent with more alternatives after the change. Only in this case, WISQ2 ranks the resulting profiles in the same manner that the original profiles were ranked. Obviously, WISQ2 is weaker than ISQ, but there is no direct link with WISQ.

In the presence of AIN, this weakening is not enough to expand the valid criteria beyond those described by the Sum of Opportunities property, as shown in the following theorem.

Theorem 3.3 *If $\succsim$ satisfies ANON, ASM, MON, AIN and WISQ2, then it also satisfies the Sum of Opportunities property.*

However, once we have chosen the Cardinality-based Criterion as an indicator of the goodness of the opportunity sets, AIN appears to lose its appeal. Eliminating AIN allows us to escape from the result of Theorem 3.3 (equivalently, substituting WISQ2 for ISQ allows us to escape from the result of Theorem 3.1). Consider the following family of criteria that combine, using a parameter α, the number of opportunities available to the individual with fewer oportunities and the number available to the other.

Definition 3.5 *A ranking $\succsim_\alpha$ on L^2 is a* Weighted Welfare Criterion *with parameter $\alpha \in [-1, 1]$ if it can be expressed as follows: for all $O, U \in L^2$,*

$$O \succsim_\alpha U \Leftrightarrow |O_{\sigma(2)}| + \alpha \cdot |O_{\sigma(1)}| \geq |U_{\sigma(2)}| + \alpha \cdot |U_{\sigma(1)}|.$$

The Weighted Welfare Criteria evaluate an opportunity distribution by means of a weighted sum of the number of opportunities of the agent with less alternatives and the number of the opportunities of the other agent. We also propose other families representing lexicographic refinements of the Weighted Welfare Criteria, using the extreme cases as tie-breakers whenever the Weighted Welfare Criterion of a given parameter α ranks two profiles as indifferent.[5]

Definition 3.6 *A ranking* $\succsim^1_\alpha$ *on* L^2 *is a* type 1 Weighted-Lexicographic Welfare Criterion *with parameter* $\alpha \in [-1, 1]$ *if it can be expressed as follows: for all* $O, U \in L^2$,

$$O \succsim^1_\alpha U \Leftrightarrow O \succ_\alpha U \ \text{or}\ [O \sim_\alpha U \ \text{and}\ O \succsim_{\alpha=-1} U].$$

Definition 3.7 *A ranking* $\succsim^2_\alpha$ *on* L^2 *is a* type 2 Weighted-Lexicographic Welfare Criterion *with parameter* $\alpha \in [-1, 1]$ *if it can be expressed as follows: for all* $O, U \in L^2$,

$$O \succsim^2_\alpha U \Leftrightarrow O \succ_\alpha U \ \text{or}\ [O \sim_\alpha U \ \text{and}\ O \succsim_{\alpha=1} U].$$

The families described above implement many criteria found in the literature. The Weighted Welfare Criterion with a parameter value of $\alpha = -1$ represents the Cardinality Difference Criterion of Kranich [5], which focuses only on equality of opportunities. At the other extreme, a value of $\alpha = 1$ represents the (*utilitarian*) sum of opportunities. Note also that $\succsim_{\alpha=1}$ coincides with $\succsim_{\gamma=1}$. For an intermediate $\alpha = 0$, we obtain the Maximin Criterion (which is discussed and generalized in Bossert et al. [1] and comes from the analysis of utility distributions; see also footnotes 3 and 4). Therefore, the value of α can also be interpreted as a reflection of our concern for efficiency.

With respect to the lexicographic versions of our criteria, notice for instance that the Leximin Criterion (translated to the opportunity context in a

[5]These families are parallel to those defined in the previous subsection. The comment on the use of other criteria as tie-breakers does also apply here in a parallel way.

natural way) is the type 2 Weighted-Lexicographic Criterion with parameter $\alpha = 0$.

Axioms ANON, ASM, MON and WISQ2 characterize these families.

Theorem 3.4 $\succsim$ *satisfies ANON, ASM, MON and WISQ2 if and only if there exists* $\alpha \in [-1, 1]$ *such that* $\succsim \in \{\succsim_\alpha, \succsim_\alpha^1, \succsim_\alpha^2\}$.

We also show that the axioms used in the characterization results (Theorems 3.2 and 3.4) are independent.

Proposition 3.1 *The following statements hold:*

1. *Axioms ANON, ASM, MON, AIN and WISQ are independent.*

2. *Axioms ANON, ASM, MON and WISQ2 are independent.*

4 The n-agent case

In this section, we will extend the results of the previous section to the general case of an arbitrary set of agents. First, we will propose and discuss suitable extensions of the axioms considered in Section 3. Second, we will provide a group of results that partially extend the theorems of the previous section. A brief explanation of this analysis will conclude the section.

Anonymity (ANON^n): $O \sim \sigma(O)$ for all $O \in L^n$.

This extension seems very natural. This is less simple for the other properties.

Assimilation (ASM^n): For all $O \in L^n$, all $x \in (X \setminus O^{\cup})$ and all $y_1, \ldots, y_n \in X$, we have that $(O_1 \cup \{x\}, \ldots, O_n \cup \{x\}) \succsim (O_1 \cup \{y_1\}, \ldots, O_n \cup \{y_n\})$ whenever:

1. $y_i \in O_i$ for all $i \in I$

or

2. for all $K \subseteq I$, there exists $y \in \{y_k\}_{k \in K}$ such that $y \notin \bigcup_{k \in K} O_k$.

The extension of part 1 is obvious. With respect to part 2, one might consider the following restriction: $y_i \notin O_i$ for all $i \in I$ and there exists $j \in I$ such that $y_j \notin O^{\cup}$. We adopt a stronger restriction (and thus a weaker axiom) to avoid establishing a fixed answer in some dubious comparisons, such as the following:

Example 4.1 Let n be an even number. Consider the profile $O \in L^n$ where $O_i = \{a_1\}$ for all $i \leq \frac{n}{2}$ and $O_i = \{a_2\}$ otherwise, with $a_1 \neq a_2$. We would like to study the following two expansions. In the first case, we consider the addition of an exogenous and common alternative x to all agents. This leads to the social distribution $U \in L^n$ where $U_i = \{a_1, x\}$ for all $i \leq \frac{n}{2}$ and $U_i = \{a_2, x\}$ otherwise, where there are two $\frac{n}{2}-$sized groups of equally treated individuals and an alternative common to both of them. In the second case, we consider the addition of a_2 to the first $\frac{n}{2}$ individuals, x to the n-th individual and a_1 to the rest, leading to the profile $V = (\{a_1, a_2\}, \{a_1, a_2\}, \ldots, \{a_1, a_2\}, \{x, a_2\})$. In this profile, there are again two classes of agents all the members of which are equally treated, again with an alternative common to all of them. The dimmensions of the groups are different, however, and to find a conclusive answer to the question of how to compare these two profiles involves some debate on polarization which will require more careful insight (see, for instance, Esteban and Ray [2] in an income framework). Notice that the extension of ASM we have adopted does not impose any comparison between U and V. However, the other *natural* extension tilts the balance weakly in favor of U.

In any case, all of these versions of Assimilation (in the n-agent case) are weaker than the version employed in Kranich [5] and Weymark [13] in which only $y_i \notin O_i$ for all $i \in I$ is required. Their version of this axiom not only compares the profiles in the previous example, but it also has some counterintuitive implications, as comented in Section 3. A similar example can be constructed for the general case.

Example 4.2 Consider the profiles O and U in Example 4.1 and the following profile $W = (\{a_1, a_2\}, \{a_1, a_2\}, \{a_1, a_2\}, \dots, \{a_1, a_2\})$, which can be obtained by expanding O through the addition of a_2 to the first $\frac{n}{2}$ agents and a_1 to the rest. W is clearly an egalitarian profile, whereas U, which would be weakly ranked above W assuming the strongest version of assimilation, is not. It seems to us that no such clear judgement should be made from the very beginning. Neither of the two versions discussed previously would compare U and W.

With respect to Monotonicity, we adopt the following version of this axiom:

Monotonicity (MON^n): For all $O \in \mathcal{N}$, for all $i, j \in I$ such that $j > i$ and for all $x \notin O_j$,

$$(O_1, \dots, O_i, \dots, O_j \cup \{x\}, \dots, O_n) \succsim (O_1, \dots, O_i \cup \{x\}, \dots, O_j, \dots, O_n),$$
$$\text{with strict preference if } j = n \text{ and } x \in O_k \text{ for all } k \neq j.$$

The first part of Monotonicity extends the two-person property by assuming that it is socially preferable to expand (with the same alternative) the opportunity set of a more disadvantaged individual. This conclusion can only be applied whenever the entire profile is nested. This constitutes a relaxation of the Generalized Pigou-Dalton Transfer Principle of Weymark [13], which applies to any profile in which O_j is contained in O_i.

A natural extension of the strict part might require removing the requirement that $j = n$. We adopt this condition as the weakest starting point, where only the addition of alternatives to the least-advantaged individual denotes a clear improvement, whereas any other expansion must be subjected to closer scrutiny (analysis of the entire social distribution and equity-efficiency issues are at stake). In Kranich [5], the axiom of Progressivity establishes this requirement for any $j < \frac{n+1}{2}$.

AIN appears to have a very natural extension. We restrict the property to apply only to cases in which the alternatives added to the pair of agents involved are equivalent with respect to the opportunity sets of the remaining individuals.

Absence of Information in Non-Nested Profiles (AIN^n): For all $O \in L^n$, for all $i, j \in I$ such that $O_i \nsubseteq O_j$ and $O_j \nsubseteq O_i$, and for all $x_i \notin O_i$, $x_j \notin O_j$, such that $x_i \in O_j \Leftrightarrow x_j \in O_i$ and $x_i \in O_k \Leftrightarrow x_j \in O_k$ for all $k \in (I \setminus \{i, j\})$,

$$(O_1, \dots, O_i \cup \{x_i\}, \dots, O_j, \dots, O_n) \sim (O_1, \dots, O_i, \dots, O_j \cup \{x_j\}, \dots, O_n).$$

The extension of the axiom of Independence of Status Quo is the following:

Independence of Status Quo (ISQ^n): For all $O, U \in \mathcal{N}$ and for all $A_i, B_i \in (L \cup \emptyset)$ such that $A_i \subseteq (O_i \cap U_i)$, $B_i \cap (O_i \cup U_i) = \emptyset$ for all $i \in I$ and $f(\cdot, O), f(\cdot, U) \in L^n$ whenever

$$f_j(\cdot, O) \not\subset f_i(\cdot, O) \text{ and } f_j(\cdot, U) \not\subset f_i(\cdot, U) \text{ for all } j < i,$$

it must follow that $O \succsim U \Rightarrow f(\cdot, O) \succsim f(\cdot, U)$.

As in ISQ, ISQ^n requires some conditions in order to rank two equally modified profiles as the original ones were ranked. These conditions are equivalent to those of the two-person case for every pair of agents. In particular, it is required that no pair of agents reverses its relative position according to the partial inclusion ordering. In a very similar fashion, the weaker versions of this property are extended.

Weak Independence of Status Quo (WISQ^n): For all $O, U \in \mathcal{N}$ and for all $A_i, B_i \in (L \cup \emptyset)$ such that $A_i \subseteq (O_i \cap U_i)$, $B_i \cap (O_i \cup U_i) = \emptyset$ for all $i \in I$ and $f(\cdot, O), f(\cdot, U) \in L^n$ whenever

$f_j(\cdot, O) \not\subset f_i(\cdot, O)$ and $f_j(\cdot, U) \not\subset f_i(\cdot, U)$ for all $j < i$

and

$(O_i \cap O_j) \setminus [f_i(\cdot, O) \cap f_j(\cdot, O)] = (U_i \cap U_j) \setminus [f_i(\cdot, U) \cap f_j(\cdot, U)]$ and $[f_i(\cdot, O) \cap f_j(\cdot, O)] \setminus (O_i \cap O_j) = [f_i(\cdot, U) \cap f_j(\cdot, U)] \setminus (U_i \cap U_j)$, for all $i, j \in \{1, \ldots, n\}$,

it must follow that $O \succsim U \Rightarrow f(\cdot, O) \succsim f(\cdot, U)$.

As in the case of two agents, WISQ^n requires stronger conditions than ISQ^n for a change not to modify the original ranking. For each pair of agents, the opportunities that have disappeared from their common set must be the same in both profiles and the same must occur with the new opportunities incorporated to their common set.

Weak Independence of Status Quo 2 (WISQ2^n): For all $O, U \in \mathcal{N}$ and for all $A_i, B_i \in (L \cup \emptyset)$ such that $A_i \subseteq (O_i \cap U_i)$, $B_i \cap (O_i \cup U_i) = \emptyset$ for all $i \in I$ and $f(\cdot, O), f(\cdot, U) \in L^n$ whenever

$$|f_j(\cdot, O)| \geq |f_i(\cdot, O)| \text{ and } |f_j(\cdot, U)| \geq |f_i(\cdot, U)| \text{ for all } j < i,$$

it must follow that $O \succsim U \Rightarrow f(\cdot, O) \succsim f(\cdot, U)$.

In WISQ2^n, the disadvantaged agent in a pair does not become the agent with more alternatives in this pair after the change.

We conclude with an extra axiom. As we have noted, the first part of the Monotonicity axiom refers to the *transfer* property, i.e., the transfer of one alternative from an advantaged individual to a more disadvantaged one constitutes a weak improvement. The following property requires the transfer property to be applied consistently.

Consistency in Transfers (CT^n): For all $O \in L^n$, for all $i, j \in I$ such that $O_i \subset O_j$, and for all $x, y \in X$ such that $x \in (O_j \setminus O_i)$ and $y \notin O^{\cup}$,

$$(O_1, \ldots, O_i, \ldots, O_j \cup \{y\}, \ldots, O_n) \sim (O_1, \ldots, O_i \cup \{x\}, \ldots, (O_j \setminus \{x\}) \cup \{y\}, \ldots, O_n) \Rightarrow O \sim (O_1, \ldots, O_i \cup \{x\}, \ldots, O_j \setminus \{x\}, \ldots, O_n).$$

Suppose that there exists a profile for which the transfer of an alternative x from an advantaged individual j to a more disadvantaged one, i, is a matter of social indifference. Suppose also that the advantaged individual j has an exclusive alternative y, i.e., an alternative not possessed by any other individual. If we remove this alternative y, making i and j less unequal, the same transfer of x from j to i can not be a strict improvement.

Some further notation is required to formulate the results of this section. We denote $\bigcap_k O = \{x \in X$ such that there exists $K \subseteq I$ with $|K| = k$ and $x \in O_i$ for all $i \in K\}$. For example, $\bigcap_n O = O^{\cap}$ and $\bigcap_1 O = O^{\cup}$. We also define $D = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ such that $1 \geq x_{i+1} \geq x_i$ for all $i \in \{1, \ldots, n-2\}$ and $\sum_{i=1}^{n-1} x_i \geq -1\}$. Consider the following natural extension of the Sum of Opportunities property.

Definition 4.1 *A ranking $\succsim$ on L^n satisfies the Sum of Opportunities property when for all $O, U \in L^n$, $|O_1| + \ldots + |O_n| > |U_1| + \ldots + |U_n| \Rightarrow O \succ U$.*

We will summarize all the results of this section in the following theorem.

Theorem 4.1 *If $\succsim \subseteq L^n \times L^n$, then the following statements hold:*

1. *If $\succsim$ satisfies ANON^n, ASM^n, MON^n and ISQ^n, then it also satisfies the Sum of Opportunities property.*

2. *If $\succsim$ satisfies ANON^n, ASM^n, MON^n, AIN^n, WISQ^n and CT^n, then there exists $\vec{\gamma} \in D$ such that for all $O, U \in L^n$,*

$$|\bigcap_n O| + \gamma_{n-1} \cdot |\bigcap_{n-1} O| + \ldots + \gamma_1 \cdot |\bigcap_1 O| >$$
$$|\bigcap_n U| + \gamma_{n-1} \cdot |\bigcap_{n-1} U| + \ldots + \gamma_1 \cdot |\bigcap_1 U| \Rightarrow O \succ U.$$

3. *If $\succsim$ satisfies ANON^n, ASM^n, MON^n, AIN^n and WISQ2^n, then it also satisfies the Sum of Opportunities property.*

4. *If $\succsim$ satisfies ANON^n, ASM^n, MON^n and WISQ2^n, then there exists $\vec{\alpha} \in D$ such that for all $O, U \in L^n$,*

$$|O_{\sigma(n)}| + \alpha_{n-1} \cdot |O_{\sigma(n-1)}| + \ldots + \alpha_1 \cdot |O_{\sigma(1)}| >$$
$$|U_{\sigma(n)}| + \alpha_{n-1} \cdot |U_{\sigma(n-1)}| + \ldots + \alpha_1 \cdot |U_{\sigma(1)}| \Rightarrow O \succ U.$$

Parts 1 and 3 are direct extensions of Theorems 3.1 and 3.3. Part 2 is a partial extension of the corresponding two-person result (Theorem 3.2), in which the new axiom of Consistency in Transfers is added. In the $n-$person version of this result, a weighted sum of intersections across dimensions serves as the main indicator of the strict preference between distributions. These weights have to satisfy a monotonicity condition: those alternatives shared by a larger number of individuals have a greater weight. Moreover, the sum of all weights must be non-negative (or, assuming that the weight of the dimension n is 1, the weights from dimensions 1 to $n-1$ must be greater than or equal to -1).

Part 4 is a partial extension of Theorem 3.4. Profiles are now ranked using a weighted sum of the cardinalities of individual opportunity sets, ordered precisely by the number of opportunities. These weights must satisfy the same restrictions as in part 2, reflecting the social objective that the more disadvantaged an individual is (with respect to the Cardinality-based Criterion), the more positive treatment should receive in social decisions. This class of social rankings includes all the *generalized Gini indexes of equality of opportunities*, a family characterized in Theorem 2 of Weymark [13] and the *generalized Gini social evaluation functions*, a family characterized in Theorem 3 of Weymark [13].[6]

We have not presented total extensions of Theorems 3.2 and 3.4. One possibility would be to incorporate the Archimedean Difference Property of Kranich [5] and Weymark [13]. With this axiom, all the distributions with the same weighted sum turn out to be indifferent in both cases. In the case of cardinalities, Weymark's criteria are the only ones to satisfy this additional axiom.

[6]The strict part of these two families behaves exactly as in our part 4 with $\sum_{i=1}^{n-1} \alpha_i = -1$ and $\sum_{i=1}^{n-1} \alpha_i > -1$, respectively.

5 Concluding remarks

In this paper, we have considered the possibility of finding absolute rankings of opportunity distributions that focus exclusively on the goodness of a given change between two different profiles of opportunity sets. It strikes us as somewhat surprising that this literature is lacking in proposals that take into account the original position in society (the status quo) when judging a given change or policy. The finding of relative proposals remains an open question in this literature.

In our results, we have observed an intimate relationship between a large group of apparently different rankings. The main point of divergence among them involves two aspects. Firstly, their concern for the different degrees of *equity* and *efficiency* that apply in a distributive problem. This point is captured by parameters $\vec{\alpha}$ and $\vec{\gamma}$ in our results. Secondly, the importance of the underlying ranking of opportunity sets. The adoption of the limited partial inclusion ranking as the only relevant information is restricting, unless we also have an interest in the common opportunities. The use of the Cardinality-based Criterion, in contrast, leads us directly to a family of compromise rankings.

Appendix: Proofs of the results

Given their similar structure, we provide a joint proof of all the theorems of the two-agent case.

Proof of the theorems of Section 3

We need the following lemmas.

Lemma 5.1 *If $\succsim$ satisfies ANON, ASM and WISQ2, then $O \sim (O_1, (O_2 \setminus \{x\}) \cup \{y\})$ for all $O \in L^2$, $x \in O_2$, $y \notin O_2$.*

Proof: We divide the proof into the following cases.

1. $x \in O_1$ and $y \in O_1$. First, if $|O_2| > 1$, consider the profile $(O_1 \setminus \{x\}, O_2 \setminus \{x\})$ and applying ASM with $y_1 = x$ and $y_2 = y$, we have that $O \succsim (O_1, (O_2 \setminus \{x\}) \cup \{y\})$. In a similar way starting with the profile $(O_1 \setminus \{y\}, O_2 \setminus \{x\})$, we can obtain $(O_1, (O_2 \setminus \{x\}) \cup \{y\}) \succsim O$, and we arrive at the desired result. If $|O_2| = 1$, by ASM, $T = (\{x, y, z\}, \{x, z\}) \sim (\{x, y, z\}, \{y, z\}) = U$ with $z \notin O^{\cup}$. Given that $f(\{z\}, O_1 \setminus \{x, y\}, \{z\}, O_2 \setminus \{x\}, T) = O$ and $f(\{z\}, O_1 \setminus \{x, y\}, \{z\}, O_2 \setminus \{x\}, U) = (O_1, (O_2 \setminus \{x\}) \cup \{y\})$, applying WISQ2, we obtain $O \sim (O_1, (O_2 \setminus \{x\}) \cup \{y\})$.

2. $x \in O_1$ and $y \notin O_1$. If $|O_1| \geq |O_2|$, with the previous profiles $T, U \in L^2$ we can obtain $f(\{y, z\}, O_1 \setminus \{x\}, \{z\}, O_2 \setminus \{x\}, T) = O$ and $f(\{y, z\}, O_1 \setminus \{x\}, \{z\}, O_2 \setminus \{x\}, U) = (O_1, (O_2 \setminus \{x\}) \cup \{y\})$. Then, given that $T \sim U$, applying WISQ2, we have that $O \sim (O_1, (O_2 \setminus \{x\}) \cup \{y\})$. If $|O_1| < |O_2|$, consider the profiles $V = (\{z, a, x\}, (\{z, a\})$ and $W = (\{z, a, y\}, \{z, a\})$, with $a \notin O^{\cup}$. By case 1 we know that $V \sim (\{z, a, x\}, \{z, x\})$. By ASM, $(\{z, a, x\}, \{z, x\}) \sim (\{z, a, y\}, \{z, y\})$. Similarly, by case 1, $(\{z, a, y\}, \{z, y\}) \sim W$. Therefore, transitivity guarantees $V \sim W$. Then, we also know that $f(\{z, a\}, O_2 \setminus \{x\}, \{z, a\}, O_1, V) = (O_2, O_1)$ and $f(\{z, a\}, O_2 \setminus \{x\}, \{z, a\}, O_1, W) = ((O_2 \setminus \{x\}) \cup \{y\}, O_1)$. By WISQ2, $(O_2, O_1) \sim ((O_2 \setminus \{x\}) \cup \{y\}, O_1)$. By ANON, $O \sim (O_2, O_1)$ and $((O_2 \setminus \{x\}) \cup \{y\}, O_1) \sim (O_1, (O_2 \setminus \{x\}) \cup \{y\})$. By transitivity, we arrive at the desired result.

3. $x \notin O_1$ and $y \in O_1$. The proof in this case is dual to case 2.

4. $x \notin O_1$ and $y \notin O_1$. If $O_1 \not\subseteq O_2$ there exists $z \in (O_1 \setminus O_2)$ such that, applying case 3, $O \sim (O_1, (O_2 \setminus \{x\}) \cup \{z\})$. And, by case 2, $(O_1, (O_2 \setminus \{x\}) \cup \{z\}) \sim (O_1, (O_2 \setminus \{x\}) \cup \{y\})$. Hence, by transitivity, $O \sim (O_1, (O_2 \setminus \{x\}) \cup \{y\})$. If $O_1 \subseteq O_2$, let $w \in O_1$ and by case 2 and ANON, we have that $O \sim ((O_1 \setminus \{w\}) \cup \{y\}, O_2)$. By case 3, it follows

that $((O_1 \setminus \{w\}) \cup \{y\}, O_2) \sim ((O_1 \setminus \{w\}) \cup \{y\}, (O_2 \setminus \{x\}) \cup \{y\})$. Finally, by case 1 and ANON, $((O_1 \setminus \{w\}) \cup \{y\}, (O_2 \setminus \{x\}) \cup \{y\}) \sim (O_1, (O_2 \setminus \{x\}) \cup \{y\})$. Applying transitivity, we conclude the proof.

□

Now, we define $Q = \{(a, b) \in \mathbb{N}^2$ such that $a \geq b\}$. With this definition, we can formulate the following lemma.

Lemma 5.2 *If $\succsim$ satisfies ANON, ASM and WISQ2, then there exists a complete preorder R over Q such that $O \succsim U$ if and only if $(|O_{\sigma(1)}|, |O_{\sigma(2)}|)\, R\, (|U_{\sigma(1)}|, |U_{\sigma(2)}|)$.*

Proof: We construct R as follows: for all $(a, b), (c, d) \in Q$, $(a, b)\, R\, (c, d) \Leftrightarrow$ there exist $O, U \in L^2$ such that $|O_{\sigma(1)}| = a, |O_{\sigma(2)}| = b, |U_{\sigma(1)}| = c, |U_{\sigma(2)}| = d$ and $O \succsim U$. First, we show that R is well-defined. Given that $\succsim$ is a complete preorder, it is sufficient to show that two profiles V, W with the same associated vector (a, b) are indifferent. By ANON, without loss of generality, $|V_1| = |W_1|$. By repeated applications of Lemma 5.1, it must be $V \sim (V_1, W_2)$. By ANON, it follows that $(V_1, W_2) \sim (W_2, V_1)$. Applying again Lemma 5.1, we obtain $(W_2, V_1) \sim (W_2, W_1)$. Finally, ANON and transitivity imply $V \sim W$. Transitivity and completeness of R can be deduced from these properties of $\succsim$. □

For the rest of the lemmas, we define the sets $Q^* = \{(a, b) \in [(\mathbb{N} \times \mathbb{N}^*) \setminus \{(1, 0)\}]$ such that $a \geq b\}$ and $Q' = \{(a, b, c) \in [(\mathbb{N}^*)^3 \setminus (\mathbb{N}^* \times \{0\} \times \{0\})]$ such that $a \geq b\}$.

Lemma 5.3 *If $\succsim$ satisfies ANON, ASM, AIN and WISQ, then there exists a complete and transitive binary relation R' over Q' such that $O \succsim U$ if and only if $(|O_{\sigma(1)} \setminus O_{\sigma(2)}|, |O_{\sigma(2)} \setminus O_{\sigma(1)}|, |O^{\cap}|)\, R'\, (|U_{\sigma(1)} \setminus U_{\sigma(2)}|, |U_{\sigma(2)} \setminus U_{\sigma(1)}|, |U^{\cap}|)$.*

Proof: Define R' in the following way: for all $(a, b, c), (d, e, f) \in Q'$, let $(a, b, c)\, R'\, (d, e, f)$ if and only if there exist $O, U \in L^2$ with $(|O_{\sigma(1)} \setminus$

$O_{\sigma(2)}|, |O_{\sigma(2)} \setminus O_{\sigma(1)}|, |O^{\cap}|) = (a, b, c)$, $(|U_{\sigma(1)} \setminus U_{\sigma(2)}|, |U_{\sigma(2)} \setminus U_{\sigma(1)}|, |U^{\cap}|) = (d, e, f)$ and $O \succsim U$.

To see that R' is well-defined, given that $\succsim$ is a complete preorder, it is sufficient to show that two profiles V and W with the same associated vector (a, b, c) are indifferent.

By ANON, without lost of generality, we suppose that $|V_2| \leq |V_1|$ and $|W_2| \leq |W_1|$. First, consider $b = 0$. That is, $V, W \in \mathcal{N}$. Then, $V = (V_2 \cup A, V_2), W = (W_2 \cup B, W_2)$ with $|V_2| = |W_2| = c$ and $|A| = |B| = a$. Consider the following profiles: $V' = (V_2 \cup A \cup \{x\}, V_2 \cup \{x\})$ and $W' = (W_2 \cup B \cup \{x\}, W_2 \cup \{x\})$, with $x \notin (V^{\cup} \cup W^{\cup})$. By repeated applications of ASM, if necessary, starting with profile $(A \cup \{x\}, \{x\})$ and transitivity it follows that $V' \sim (W_2 \cup A \cup \{x\}, W_2 \cup \{x\})$. If $a = 0$ or $A = B$, $V' \sim W'$. Otherwise, let $M = (A \cup \{x, t\}, \{x, t\})$ and $N = (B \cup \{x, t\}, \{x, t\})$ with $t \notin (V^{\cup} \cup W^{\cup} \cup \{x\})$. Then, we have by ASM starting with profile $(A \cup \{x\}, \{x\})$ that $M \succsim (A \cup \{x, t\}, \{x, y\})$ with $y \in (A \setminus B)$. Also, by ASM starting with profile $((A \setminus \{y\}) \cup \{x, t\}, \{x\})$, we have that $(A \cup \{x, t\}, \{x, y\}) \succsim (A \setminus \{y\} \cup \{x, t, z\}, \{x, t\})$ with $z \in (B \setminus A)$. Repeating this process for all the alternatives in $(A \setminus B)$ and $(B \setminus A)$, it follows that $M \succsim N$. Similarly, $N \succsim M$. Thus, $M \sim N$. Because $f(\{t\}, W_2, \{t\}, W_2, M) = (W_2 \cup A \cup \{x\}, W_2 \cup \{x\})$ and $f(\{t\}, W_2, \{t\}, W_2, N) = W'$, and noticing that the conditions for WISQ hold, the application of WISQ implies $(W_2 \cup A \cup \{x\}, W_2) \sim W'$. Transitivity guarantees that, in any case, $V' \sim W'$. Then, given that $f(\{x\}, \emptyset, \{x\}, \emptyset, V') = V$ and $f(\{x\}, \emptyset, \{x\}, \emptyset, W') = W$ and the conditions for WISQ hold, it must be that $V \sim W$.

Second, let $b \neq 0$. If $c = 0$, assume without loss of generality that $a > 1$ (otherwise, the application of ANON is enough to establish the result) and $(V^{\cup} \cap W^{\cup}) = \emptyset$. Then, we construct the following profiles: $P = (V_1 \cup \{x\}, \{x\})$ and $Q = (W_1 \cup \{x\}, \{x\})$, with $x \notin (V^{\cup} \cup W^{\cup})$. We know, by the previous reasoning, that $P \sim Q$. Then, we have that $f(\{x\}, \emptyset, \{x\}, V_2, P) = V$ and $f(\{x\}, \emptyset, \{x\}, V_2, Q) = (W_1, V_2)$. Given that the conditions of WISQ are satisfied, we can establish that $V \sim (W_1, V_2)$. Now, consider a set $C \subseteq W_1$ such that $|C| = (a-b)$. Then, we can apply AIN $(a-b)$ times in a convenient way to the starting profile $(W_1 \setminus C, V_2)$ to obtain, jointly with transitivity, that $(W_1 \setminus C, V_2 \cup C) \sim (W_1, V_2)$. Then, we can apply the former reasoning to $(W_1 \setminus C, V_2 \cup C)$ and we obtain $(W_1 \setminus C, V_2 \cup C) \sim (W_1 \setminus C, W_2 \cup C)$. Applying AIN another $(a-b)$ times to the starting profile $(W_1 \setminus C, W_2)$, we obtain $(W_1 \setminus C, W_2 \cup C) \sim W$. Transitivity guarantees that $V \sim W$. If $c \neq 0$, by ASM we have that $V \sim ((V_1 \setminus V_2) \cup W^\cap, (V_2 \setminus V_1) \cup W^\cap)$. The rest of the proof follows a similar argument as in the case $c = 0$.

Transitivity and Completeness of R' are obvious from the transitivity and completeness of $\succsim$ and the fact that every element in Q' can be associated with some profile in L^2. □

Lemma 5.4 *If $\succsim$ satisfies ANON, ASM, AIN and WISQ, then there exists a complete preorder R^* over Q^* such that $O \succsim U$ if and only if $(|O^\cup|, |O^\cap|)\, R^*\, (|U^\cup|, |U^\cap|)$.*

Proof: Define R^* by: $(a, b)\, R^*\, (c, d)$ if and only if there exist $O, U \in L^2$ with $(|O^\cup|, |O^\cap|) = (a, b)$, $(|U^\cup|, |U^\cap|) = (c, d)$ and $O \succsim U$.

To see that R^* is well-defined, given that $\succsim$ is a complete preorder, it is sufficient to show that two profiles V and W with the same associated vector (a, b) are indifferent. Without loss of generality, by Lemma 5.3 and ANON, we assume that $V^\cup = W^\cup$, $V^\cap = W^\cap$, $V = \sigma(V)$, $W = \sigma(W)$ and $W_1 \subseteq V_1$. If $V = W$, reflexivity applies. Otherwise, we have $(V_1 \setminus W_1) \neq \emptyset$.

First, suppose that $V, W \notin \mathcal{N}$. If $|V_1| = 1$, it must be that $V = W$, and reflexivity guarantees that $V \sim W$. If $|V_1| > 1$, consider the profile $(V_1 \setminus \{x\}, V_2)$ with $x \in (V_1 \setminus W_1)$. By AIN, separately adding x to both agents in this profile, we obtain $V \sim (V_1 \setminus \{x\}, V_2 \cup \{x\})$. Repeating this process, and applying transitivity, we obtain $V \sim W$. Second, suppose that $V \in \mathcal{N}$. Therefore, $V^\cap$ is non-empty. Consider the profile $V' = (V_1 \setminus \{x\}, V_2)$

with $x \in V^{\cap}$. Select $a \in (V_1 \setminus V_2)$. Because $x \in V_2'$ and $a \in V_1'$ we can apply AIN, obtaining $V \sim (V_1 \setminus \{x\}, V_2 \cup \{a\})$. This new profile is not nested and the former analysis holds. Transitivity then implies that $V \sim W$.

Transitivity and Completeness of R^* are obvious from the transitivity and completeness of $\succsim$ and the fact that every element in Q^* can be associated with some profile in L^2. □

Lemma 5.2 ensures that if a criterion satisfies ANON, ASM and WISQ2, we can focus exclusively on (Q, R). In a similar way, when satisfying ANON, ASM, AIN and WISQ, Lemma 5.4 ensures that it is enough to work with (Q^*, R^*).

We describe some properties on (Q, R) and (almost dual conditions) on (Q^*, R^*). The symmetric and asymmetric parts of R and R^* are defined as usual. All of these properties are used in the proof of the theorem.

(1): $(a+1, b+1)\, R\, (a, b)$ for all $(a, b) \in Q$.

(2): $(a, b+1)\, R\, (a+1, b)$ for all $(a, b) \in Q$ such that $a \neq b$.

(3): $(a, b+1)\, P\, (a, b)$ for all $(a, b) \in Q$ such that $a \neq b$.

(4): For all $e, f \in \mathbb{Z}$ and all $(a, b), (c, d), (a+e, b+f), (c+e, d+f) \in Q$,

$$(a, b)\, R\, (c, d) \Rightarrow (a+e, b+f)\, R\, (c+e, d+f).$$

(1*): $(a+1, b+1)\, R^*\, (a, b)$ for all $(a, b) \in Q^*$.

(2*): $(a+1, b+1)\, R^*\, (a+2, b)$ for all $(a, b) \in Q^*$.

(3*): $(a, b+1)\, P^*\, (a, b)$ for all $(a, b) \in Q^*$ such that $a \neq b \neq 0$.

(4*): For all $e, f \in \mathbb{Z}$ and all $(a, b), (c, d), (a+e, b+f), (c+e, d+f) \in Q^*$ such that $b \neq 0$ and $d \neq 0$,

$$(a, b)\, R^*\, (c, d) \Rightarrow (a+e, b+f)\, R^*\, (c+e, d+f).$$

Given Lemma 5.2, we have that the axioms have the following implications on (Q, R): ASM implies (1), MON implies (2) and (3), and WISQ2 implies (4).

In the same way, by Lemma 5.4, we have that some axioms of the theorem have implications on (Q^*, R^*). They are the following: ASM implies (1^*) and (2^*), MON implies (3^*) and WISQ implies (4^*).

The design of the proofs is the following. We will start with Theorem 3.4. The results concerning the Sum of Opportunities property (Theorems 3.1 and 3.3) can be easily deduced from this theorem. Finally, we prove 3.2 by analogy.

The following list of claims will help us to derive our first result.

Claim 5.1 *If R satisfies (4), then for all $(a, b), (c, d) \in Q$ and all $e, f \in \mathbb{Z}$ such that $(a + e, b + f), (c + e, d + f) \in Q$, we have that*

$$(a + e, b + f)R(a, b) \Leftrightarrow (c + e, d + f)R(c, d).$$

Proof: Using the definition of (4), this claim is trivial. □

Therefore, we may establish a non-empty correspondence $h : \mathbb{Z}^2 \to \{G, B\}$ such that for all $(x_1, x_2) \in \mathbb{Z}^2$, we have that

$$G \in h[(x_1, x_2)] \Leftrightarrow (a + x_1, b + x_2)\, R\, (a, b) \text{ for all } (a, b), (a + x_1, b + x_2) \in Q$$
$$B \in h[(x_1, x_2)] \Leftrightarrow (a, b)\, R\, (a + x_1, b + x_2) \text{ for all } (a, b), (a + x_1, b + x_2) \in Q.$$ [7]

Claim 5.2 *If R satisfies (4), then for all $(x_1, x_2) \in \mathbb{Z}^2$,*

$$G \in h[(x_1, x_2)] \Leftrightarrow B \in h[(-x_1, -x_2)].$$

Proof: Let $(x_1, x_2) \in \mathbb{Z}^2$ be such that $G \in h[(x_1, x_2)]$. That is, $(a + x_1, b + x_2)\, R\, (a, b)$ for any $(a, b) \in Q$. Then, by definition, $B \in h[(-x_1, -x_2)]$. □

Hence, it is sufficient to consider the elements of $(\mathbb{N}^* \times \mathbb{Z})$ instead of Q.

[7] Completeness of R guarantees the non-emptyness of h, whereas the fact that X is infinite ensures that h has a full domain. Notice, however, that h is not necessarily uniquely-valued.

Claim 5.3 *If R satisfies (4), then for all $(x_1, x_2) \in (\mathbb{N}^* \times \mathbb{Z})$ and all $k \in \mathbb{N}$,*

$$h[(x_1, x_2)] = h[(kx_1, kx_2)].$$

Proof: First, select $(a, b) \in Q$ such that $(a + kx_1, b + kx_2) \in Q$. Suppose that $G \in h[(x_1, x_2)]$. That is, $(a + x_1, b + x_2)\, R\, (a, b)$. Then, by Claim 5.1 we can apply the same change to $(a + x_1, b + x_2)$ and we have that $(a + 2x_1, b + 2x_2)\, R\, (a + x_1, b + x_2)$ and, applying transitivity, we have that $(a + 2x_1, b + 2x_2)\, R\, (a, b)$. Repeating this process, we have that $(a + kx_1, b + kx_2)\, R\, (a, b)$. Therefore, $G \in h[(kx_1, kx_2)]$.

Now, consider $G \in h[(kx_1, kx_2)]$ and suppose that $G \notin h[(x_1, x_2)]$. Then, by completeness, $(a, b)\, P\, (a + x_1, b + x_2)$ and, applying Claim 5.1, $(a + x_1, b + x_2)\, P\, (a + 2x_1, b + 2x_2)$. Following the same procedure as above and applying transitivity, we conclude that $(a, b)\, P\, (a + kx_1, b + kx_2)$. Thus, $G \notin h[(kx_1, kx_2)]$, contradicting the assumption.

The proof is similar for B. □

Claim 5.4 *If R satisfies (4), then for all $(x_1, x_2), (y_1, y_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} = \frac{y_2}{y_1}$, we have:*

$$h[(x_1, x_2)] = h[(y_1, y_2)].$$

Proof: Let $(x_1, x_2), (y_1, y_2) \in (\mathbb{N} \times \mathbb{Z})$ be such that $\frac{x_2}{x_1} = \frac{y_2}{y_1}$ and suppose that $G \in h[(x_1, x_2)]$. Then, $x_1 = \frac{p}{q} y_1$ and $x_2 = \frac{p}{q} y_2$, with $p, q \in \mathbb{N}$. Given that $G \in h[(x_1, x_2)]$, we know by Claim 5.3 that $G \in h[(qx_1, qx_2)] = h[(py_1, py_2)]$. Applying Claim 5.3 to (y_1, y_2), we conclude that $G \in h[(y_1, y_2)]$. The proof is similar for B. □

Claim 5.5 *If R satisfies (3) and (4), then for all $(x_1, x_2), (y_1, y_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} < \frac{y_2}{y_1}$,*

$$G \in h[(x_1, x_2)] \Rightarrow h[(y_1, y_2)] = \{G\}.$$

Proof: Let $(x_1, x_2), (y_1, y_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} < \frac{y_2}{y_1}$ and $G \in h[(x_1, x_2)]$. Then, by Claim 5.3, $G \in h[(y_1 x_1, y_1 x_2)]$. Given that $x_2 y_1 < x_1 y_2$ and applying (3) we have that $h[(y_1 x_1, y_2 x_1)] = \{G\}$. Then, applying Claim 5.3, we have that $h[(y_1, y_2)] = \{G\}$. □

Claim 5.6 *If R satisfies (3) and (4), then for all $(x_1, x_2), (y_1, y_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} > \frac{y_2}{y_1}$,*

$$B \in h[(x_1, x_2)] \Rightarrow h[(y_1, y_2)] = \{B\}.$$

Proof: The proof is similar to that of Claim 5.5. □

Claim 5.7 *If R satisfies (3) and (4), then there exists $\alpha \in (\mathbb{R} \cup \{-\infty, +\infty\})$ such that for all $(a, b), (c, d) \in Q$,*

$$b + \alpha \cdot a > d + \alpha \cdot c \Rightarrow (a, b) P (c, d).$$

Proof: By the previous discussion, we can focus on $(\mathbb{N}^* \times \mathbb{Z})$. We have to prove that $x_2 + \alpha \cdot x_1 > 0 \Rightarrow h[(x_1, x_2)] = \{G\}$.

The reflexivity of R and (3) imply that $G \in h[(0, x_2)] \Leftrightarrow x_2 \geq 0$ and $B \in h[(0, x_2)] \Leftrightarrow x_2 \leq 0$, as desired. We now have to determine which elements of $(\mathbb{N} \times \mathbb{Z})$ map into G and B, respectively.

We define in $(\mathbb{R} \cup \{-\infty, +\infty\})$ (with the usual order):

$\beta_G = \inf\{\frac{x_2}{x_1}$ such that $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ is such that $G \in h[(x_1, x_2)]\}$
$\beta_B = \sup\{\frac{x_2}{x_1}$ such that $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ is such that $B \in h[(x_1, x_2)]\}$

We are going to prove that $\beta_G = \beta_B$. If $\beta_G > \beta_B$, we have that there exists $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\beta_G > \frac{x_2}{x_1} > \beta_B$. Then, $h[(x_1, x_2)] = \emptyset$, which is impossible. If $\beta_G < \beta_B$, we select $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\beta_G < \frac{x_1}{x_2} < \beta_B$. Then, by Claim 5.5, $h[(x_1, x_2)] = \{G\}$, and by Claim 5.6, $h[(x_1, x_2)] = \{B\}$, which is a contradiction. Therefore, $\beta_G = \beta_B = \beta^*$. By Claim 5.5, for all $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} > \beta^*$, $h[(x_1, x_2)] = \{G\}$. Similarly, by Claim 5.6, for all $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} < \beta^*$, $h[(x_1, x_2)] = \{B\}$. It is easy to see that the only way to implement these aspects of the ranking is the one proposed by the claim, with $\alpha = -\beta^*$. □

We prove that $\alpha \in [-1, 1]$. By (1), $G \in h[(1,1)]$, implying that $\alpha \geq -1$. By (2), $G \in h[(-1,1)]$ and by Claim 5.2, $B \in h[(1,-1)]$, implying that $\alpha \leq 1$.

Now, we proceed to complete the sufficient part of Theorem 3.4. The previous claims, with Lemma 5.2, determine the ranking to be a Weighted Welfare Criteria in all cases in which α is not rational (because Q is composed by natural numbers). If α is a rational number, we may have some profiles whose ranking has not yet been determined.

In that case, there exist elements $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} = \beta^*$. We have, given Claim 5.4, that all these elements have the same image by h. Thus, we have three possibilities:

(a) For all $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} = \beta^*$, $h[(x_1, x_2)] = \{G\}$. This case, with the results of Lemma 5.2, corresponds with the type 2 Weighted-Lexicographic Welfare Criterion with a value of $\alpha = -\beta^*$.

(b) For all $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} = \beta^*$, $h[(x_1, x_2)] = \{B\}$. This case, with the results of Lemma 5.2, corresponds with the type 1 Weighted-Lexicographic Welfare Criterion with a value of $\alpha = -\beta^*$.

(c) For all $(x_1, x_2) \in (\mathbb{N} \times \mathbb{Z})$ such that $\frac{x_2}{x_1} = \beta^*$, $h[(x_1, x_2)] = \{G, B\}$. This case, given Lemma 5.2, corresponds with the Weighted Welfare Criterion with a value of $\alpha = -\beta^*$.

This concludes the sufficient part of the proof of Theorem 3.4. It is straightforward to see that all involved rankings satisfy the axioms, thus concluding the proof.

The proof of Theorem 3.1 follows easily from the previous characterization. Let $O = (\{x, y, z\}, \{x, y\})$ and $U = (\{x, y, z, w\}, \{x\})$. We have that for all $\alpha < 1$, the Weighted Welfare Criterion and its lexicographic refinements rank O strictly over U. Consider the change in which the set $A = \{a_1, a_2, a_3\}$ is added to the disadvantaged individuals in both profiles, being $A \cap U^{\cup} = \emptyset$. These criteria rank $(\{x, y, z, w\}, \{x\} \cup A)$ strictly over $(\{x, y, z\}, \{x, y\} \cup A)$. This constitutes a violation of ISQ. However, it is obvious that for $\alpha = 1$ the Weighted Welfare Criterion and its lexicographic refinement satisfy the Sum of Opportunities. Therefore, any criterion satisfying ANON, ASM, MON and ISQ also satisfies the Sum of Opportunities property.[8]

Similarly, any ranking different from the Weighted Welfare Criterion with $\alpha = 1$ does not satisfy AIN, proving Theorem 3.3.[9]

By analogy with Theorem 3.4, the remaining result is straightforward. Given Lemma 5.4, we can focus on (Q^*, R^*) and we know that (1^*), (2^*), (3^*) and (4^*) hold. Although the domains Q and Q^* are not completely equal, it is easy to check that all the implications of (1), (2), (3) and (4) on (Q, R) are equivalent to those of (1^*), (2^*), (3^*) and (4^*) on (Q^*, R^*). This, joined with the fact that all the proposed rankings satisfy the axioms, concludes the proof of Theorem 3.2.

Proof of Proposition 3.1

Let $x, y \in X$, with $x \neq y$. Consider the following rankings:

$$O \succ_a U \Leftrightarrow \begin{cases} O \succ_{\gamma=1} U & \text{or} \\ O = (\{x\}, \{y\}) \text{ and } U = (\{y\}, \{x\}). \end{cases}$$

$$O \succsim_b U \Leftrightarrow |O^{\cap}| + 1.5 \cdot |O^{\cup}| \geq |U^{\cap}| + 1.5 \cdot |U^{\cup}|.$$

$$O \sim_c U \text{ for all } O, U \in L^2.$$

$$O \succsim_d U \Leftrightarrow |O_{\sigma(1)} \setminus O^{\cap}| \leq |U_{\sigma(1)} \setminus U^{\cap}|.$$

[8]Since $\succsim_{\alpha=1}$ and $\succsim^1_{\alpha=1}$ satisfy ISQ, the previous result is, indeed, a characterization theorem.

[9]Since $\succsim_{\alpha=1}$ satisfies AIN, the previous result is, indeed, a characterization theorem.

$O \succsim_e U \Leftrightarrow \frac{|O^\cap|}{|O^\cup|} \geq \frac{|U^\cap|}{|U^\cup|}$.

Any of them satisfies all the axioms in part 1 of the proposition, but one. Rankings $\succsim_a$, $\succsim_b$, $\succsim_c$, $\succsim_d$ and $\succsim_e$ do not satisfy ANON, ASM, MON, AIN and WISQ, respectively. Therefore, the proof of part 1 is completed.

Consider the following additional rankings:

$O \succsim_f U \Leftrightarrow |O_2| \geq |U_2|$.

$O \succsim_g U \Leftrightarrow |O_{\sigma(2)}| - 1.5 \cdot |O_{\sigma(1)}| \geq |U_{\sigma(2)}| - 1.5 \cdot |U_{\sigma(1)}|$.

As in the previous reasoning, rankings $\succsim_f$, $\succsim_g$, $\succsim_c$ and $\succsim_{\gamma=0}$ do not satisfy ANON, ASM, MON and WISQ2, respectively, thus proving part 2.

Proof of Theorem 4.1

Because the strategy of the proof is similar to that of the theorems of Section 3, we start by proving some parallel lemmas. We define $Q = \{(x_1, \ldots, x_n) \in \mathbb{N}^n$ such that $x_i \geq x_{i+1}$ for all $i \in \{1, \ldots, n-1\}\}$. We denote by $\vec{1}$ the element of Q such that all its elements are 1. That is, $\vec{1} = (1, \ldots, 1)$.

Lemma 5.5 *If $\succsim$ satisfies ANON^n, ASM^n and WISQ2^n, then $O \sim (O_1, \ldots, (O_i \setminus \{x\}) \cup \{y\}, \ldots, O_n)$ for all $O \in L^n$, $i \in \{1, \ldots, n\}$, $x \in O_i$ and $y \notin O_i$.*

Proof: By ANON^n, we assume, without loss of generality, that the profile O is ordered using the cardinality ranking, in the sense that $|O_k| \geq |O_{k+1}|$ for all $k \in \{1, \ldots, n-1\}$. We divide the proof into the following cases:

1. If $i = n$, consider the profile $U = (\{a, x\}, \ldots, \{a, x\}, \{a\})$ with $a \notin (O^\cup \cup \{y\})$. Then, applying ASM^n to the profile U, we have that $V = (\{a, x, y\}, \ldots, \{a, x, y\}, \{a, y\}) \succsim (\{a, x, y\}, \ldots, \{a, x, y\}, \{a, x\}) = W$. Following a similar reasoning starting with the profile $(\{a, y\}, \ldots, \{a, y\}, \{a\})$, we have that $W \succsim V$. Hence, $V \sim W$. Given that $V, W \in \mathcal{N}$, we can apply WISQ2^n and we have the desired result.

2. If $i = (n-1)$, we divide the proof into two cases:

   (a) If $x \in O_n$, consider the profile $U = (\{a,x\},\ldots,\{a,x\},\{a\},\{a\})$ with $a \notin (O^{\cup} \cup \{y\})$. Then, applying ASM^n to the profile U, we have that $V = (\{a,x,y\},\ldots,\{a,x,y\},\{a,y\},\{a,y\}) \succsim (\{a,x,y\}, \ldots,\{a,x,y\},\{a,x\},\{a,x\}) = W$. Similarly, starting with the profile $(\{a,y\},\ldots,\{a,y\},\{a\},\{a\})$ we have that $W \succsim V$. Hence, $V \sim W$. Given that $V, W \in \mathcal{N}$, WISQ2^n implies that $O \sim (O_1,\ldots,O_{n-2},(O_{n-1} \setminus \{x\}) \cup \{y\},(O_n \setminus \{x\}) \cup \{y\})$. Now, we can apply Case 1 and we have that $(O_1,\ldots,O_{n-2},(O_{n-1} \setminus \{x\}) \cup \{y\},(O_n \setminus \{x\}) \cup \{y\}) \sim (O_1,\ldots,O_{n-2},(O_{n-1} \setminus \{x\}) \cup \{y\},O_n)$. By transitivity, we have the desired result.

   (b) If $x \notin O_n$, we know by Case 1 that we can replace one alternative in O_n with x and the ranking remains invariant. Then, we can apply the same reasoning as in (a) to reach the desired conclusion.

3. If $i < (n-1)$, we follow the iterative procedure described in Case 2.

□

Lemma 5.6 *If $\succsim$ satisfies ANON^n, ASM^n and WISQ2^n, then there exists a complete preorder R over Q such that $O \succsim U$ if and only if $(|O_{\sigma(1)}|,\ldots, |O_{\sigma(n)}|)\, R\, (|U_{\sigma(1)}|,\ldots,|U_{\sigma(n)}|)$.*

Proof: We construct R as follows: for all $\vec{a},\vec{b} \in Q$, $\vec{a}R\vec{b} \Leftrightarrow$ there exist $O, U \in L^n$ such that $(|O_{\sigma(1)}|,\ldots,|O_{\sigma(n)}|) = \vec{a}$, $(|U_{\sigma(1)}|,\ldots,|U_{\sigma(n)}|) = \vec{b}$ and $O \succsim U$. The proof follows easily from Lemma 5.5 and ANON using an argument similar to that used to prove Lemma 5.2. □

Lemma 5.7 *Let $i, j \in I$. If $\succsim$ satisfies ANON^n, AIN^n and CT^n, if $O \in L^n$ is such that $O_i \subset O_j$, and if there exist $\hat{y},\hat{z} \in X$ such that $\hat{y} \in O_i$, $\hat{z} \in (O_j \setminus O_i)$ and $\hat{y} \in O_k \Leftrightarrow \hat{z} \in O_k$ for all $k \in (I \setminus \{i,j\})$, then for all $x \notin O_j$,*

$$(O_1,\ldots,O_i \cup \{x\},\ldots,O_j,\ldots,O_n) \sim (O_1,\ldots,O_i,\ldots,O_j \cup \{x\},\ldots,O_n)$$

Proof: We decompose the proof into two cases.

1. If $|O_j \setminus O_i| > 1$, consider the profile $(O_1, \ldots, O_i \cup \{\hat{z}\}, \ldots, O_j \setminus \{\hat{y}\}, \ldots, O_n)$. Now, we apply AIN^n to this profile and we have that $M = (O_1, \ldots, O_i \cup \{\hat{z}, x\}, \ldots, O_j \setminus \{\hat{y}\}, \ldots, O_n) \sim (O_1, \ldots, O_i \cup \{\hat{z}\}, \ldots, (O_j \setminus \{\hat{y}\}) \cup \{x\}, \ldots, O_n) = N$. Consider the profile $(O_1, \ldots, O_i \cup \{x\}, \ldots, O_j \setminus \{\hat{y}\}, \ldots, O_n)$. Applying AIN^n to this profile, we g that $M \sim (O_1, \ldots, O_i \cup \{x\}, \ldots, O_j, \ldots, O_n)$. Now, consider the profile $(O_1, \ldots, O_i, \ldots, (O_j \setminus \{\hat{y}\}) \cup \{x\}, \ldots, O_n)$. Applying AIN^n to this profile, we have that $N \sim (O_1, \ldots, O_i, \ldots, O_j \cup \{x\}, \ldots, O_n)$. Applying transitivity, we conclude that $(O_1, \ldots, O_i \cup \{x\}, \ldots, O_j, \ldots, O_n) \sim (O_1, \ldots, O_i, \ldots, O_j \cup \{x\}, \ldots, O_n)$.

2. If $|O_j \setminus O_i| = 1$, consider the profile $(O_1, \ldots, O_i, \ldots, O_j \cup \{w\}, \ldots, O_n)$ with $w \notin O^{\cup}$. Now, we can apply the former analysis and we have that $(O_1, \ldots, O_i \cup \{x\}, \ldots, O_j \cup \{w\}, \ldots, O_n) \sim (O_1, \ldots, O_i, \ldots, O_j \cup \{x, w\}, \ldots, O_n)$. Now, we apply CT^n and we have that $(O_1, \ldots, O_i \cup \{x\}, \ldots, O_j, \ldots, O_n) \sim (O_1, \ldots, O_i, \ldots, O_j \cup \{x\}, \ldots, O_n)$.

□

Lemma 5.8 *If $\succsim$ satisfies ANON^n, AIN^n and CT^n, then for all $O \in L^n$, all $x \notin O^{\cup}$ and all $K, K' \subseteq I$ such that $|K| = |K'|$, we have that $U \sim V$, where U is a profile in which the agents $i \in K$ have the set $O_i \cup \{x\}$ and the agents $j \in (I \setminus K)$ have the set O_j and the profile V is a profile in which the agents $i \in K'$ have the set $O_i \cup \{x\}$ and the agents $j \in (I \setminus K')$ have the set O_j.*

Proof: Note that the agents $i \in (K \cap K')$ have the alternative x and, hence, the same set of opportunities in U and V. Therefore, we focus on generic agents $i \in (K \setminus K')$ and $j \in (K' \setminus K)$. We will prove that it is a matter of indifference to add an alternative x to i or to j.

If $O_i \nsubseteq O_j$ and $O_j \nsubseteq O_i$, we can apply AIN^n to the profile O and the alternative x and we have that $(O_1, \ldots, O_i \cup \{x\}, \ldots, O_n) \sim (O_1, \ldots, O_j \cup \{x\}, \ldots, O_n)$.

If $O_i = O_j$, ANON^n implies the desired result.

If $O_i \subset O_j$, we select $y \in O_i$ and denote by $T_y \in (I \setminus \{i\})$ the set of individuals that possess y. Then, we construct the profile W in the following way: $W_j = O_j \cup \{z\}$ if $j \in T_y$ and $W_j = O_j$ otherwise, where $z \notin O^{\cup}$. Then, given the construction of the profile, we can apply Lemma 5.7 to W and we have that $(W_1, \dots, W_i \cup \{x\}, \dots, W_n) \sim (W_1, \dots, W_j \cup \{x\}, \dots, W_n)$. Now, applying CT^n (eliminating z) we arrive in the desired result.

This procedure can be applied to any pair of individuals and the successive profiles to obtain the lemma. □

Lemma 5.9 *If $\succsim$ satisfies ANON^n, ASM^n, WISQ^n, AIN^n and CT^n, then for all $O \in L^n$, all $K \subseteq I$, all $y \notin O^{\cup}$ and all $x \in X$ such that $x \in O_i$ for all $i \in K$ and $x \notin O_j$ for all $j \in (I \setminus K)$, we have that $O \sim O'$, where O' is a profile in which the agents $i \in K$ have the opportunity set $((O_i \setminus \{x\}) \cup \{y\})$ and the agents $j \in (I \setminus K)$ have the set O_j.*

Proof: By ANON^n, we assume, without loss of generality, that the profile O is ordered using the cardinality ranking, in the sense that $|O_k| \geq |O_{k+1}|$ for all $k \in \{1, \dots, n\}$. We divide the proof into the following cases:

1. If $K = I$, we know by ASM^n that $U = (\{a, x\}, \{a, x\}, \dots, \{a, x\}) \sim (\{a, y\}, \{a, y\}, \dots, \{a, y\}) = V$, where $a \notin (O^{\cup} \cup \{y\})$. Given that $U, V \in \mathcal{N}$, WISQ^n implies that $((O_1 \setminus \{x\}) \cup \{y\}, \dots, (O_n \setminus \{x\}) \cup \{y\}) \sim O$.

2. If $|K| = |I| - 1$, we divide the proof into two cases:

   (a) If $K = \{2, \dots, n\}$, consider the profile $U = (\{a, x\}, \{a\}, \dots, \{a\})$, with $a \notin (O^{\cup} \cup \{y\})$. We apply ASM^n to the profile U and we have that $V = (\{a, x, y\}, \{a, y\}, \dots, \{a, y\}) \succsim (\{a, x, y\}, \{a, x\}, \dots, \{a, x\}) = W$. Similarly, by reversing the roles of x and y in the preceding argument, we have that $W \succsim V$ and, hence, by transitivity, $V \sim W$. Given that $V, W \in \mathcal{N}$, we can apply WISQ^n and we conclude that $O \sim (O_1, (O_2 \setminus \{x\}) \cup \{y\}, \ldots, (O_n \setminus \{x\}) \cup \{y\})$.

(b) If $K \neq \{2, \ldots, n\}$, we have that $1 \in K$ and there exists $j \in \{2, \ldots, n\}$ such that $j \notin K$. If $|O_1| > 1$, by Lemma 5.8, we have that $O \sim (O_1 \setminus \{x\}, O_2, \ldots, O_j \cup \{x\}, \ldots, O_n) = W$. Note that, in profile W, the alternative x is possessed only by the individuals in $\{2, \ldots, n\}$. Then, we can apply Case (a) to the profile W and we have that $W \sim W'$, where W' is a profile such that $W_1' = W_1 = O_1 \setminus \{x\}$, $W_j' = (W_j \setminus \{x\}) \cup \{y\} = O_j \cup \{y\}$ and $W_i' = (W_i \setminus \{x\}) \cup \{y\} = (O_i \setminus \{x\}) \cup \{y\}$ for all $i \notin \{1, j\}$. Now, by Lemma 5.8, $W' \sim O'$. Hence, by transitivity, $O \sim O'$.

If $|O_1| = |O_j| = 1$, we apply ANON^n and we have that $O \sim U$, where U is a profile such that $U_1 = O_j$, $U_j = O_1$ and $U_i = O_i$ for all $i \notin \{1, j\}$. Then, in profile U, the alternative x is possessed only by the individuals in $\{2, \ldots, n\}$. Then, we can apply Case 1 and we have that $U \sim (U_1, (U_2 \setminus \{x\}) \cup \{y\}, \ldots, (U_n \setminus \{x\}) \cup \{y\})$. Given that $|U_1| = |(U_j \setminus \{x\}) \cup \{y\}| = 1$, we have by ANON^n that $(U_1, (U_2 \setminus \{x\}) \cup \{y\}, \ldots, (U_n \setminus \{x\}) \cup \{y\}) \sim O'$. Hence, by transitivity, $O \sim O'$.

If $|O_1| = 1$, but $|O_j| > 1$, consider the profile $V \in L^n$ such that $V_j = (O_j \setminus \{z\})$ and $V_i = O_i$ for all $i \neq j$, where $z \in O_j$. Then, by AIN^n, we have that $V \sim V'$, where $V_1' = O_1 \cup \{z\}$ and $V_i' = O_i$ for all $i \neq 1$. Then, we can apply AIN^n to V' and we have that $V' \sim V''$, where $V_1'' = O_1 \cup \{z\} \setminus \{x\}$, $V_j'' = O_j \setminus \{x\}$ and $V_i'' = O_i$ for all $i \notin \{1, j\}$. Then, we have that x is possessed in V'' only by the individuals in $\{2, \ldots, n\}$. Therefore, we can apply Case (a) to V'' and substitute the alternative x by y in these individuals. By a similar process of applications of AIN^n we arrive, jointly with transitivity, at $O \sim O'$.

3. If $|K| < |I| - 1$, we can apply a similar reasoning to that of Case 2. If $K = \{n+1-|K|, \ldots, n\}$ we can apply ANON^n and WISQ^n in a similar way to that of subcase (a) to obtain the desired result. If, however, $K \neq \{n+1-|K|, \ldots, n\}$, we can apply Lemma 5.8, ANON^n and AIN^n as in subcase (b) to conclude the result.

□

For the following lemma, we need to define the set $Q^* = \{(x_1, \ldots, x_n) \in [\mathbb{N} \times (\mathbb{N}^*)^{n-1}]$ such that $x_i \geq x_{i+1}$ for all $i \in \{1, \ldots, n-1\}$ and $\sum_{i=1}^{n} x_i \geq n\}$.

Lemma 5.10 *If $\succsim$ satisfies ANON^n, ASM^n, WISQ^n, AIN^n and CT^n, then there exists a complete preorder R^* over Q^* such that $O \succsim U \Leftrightarrow (|\bigcap_1 O|, |\bigcap_2 O|, \ldots, |\bigcap_n O|)\, R^*\, (|\bigcap_1 U|, |\bigcap_2 U|, \ldots, |\bigcap_n U|)$.*

<u>Proof</u>: Define R^* by:

$$(a_1, \ldots, a_n)\, R^*\, (b_1, \ldots, b_n) \text{ if and only if there exist } O, U \in L^2 \text{ with } (|\textstyle\bigcap_1 O|, |\bigcap_2 O|, \ldots, |\bigcap_n O|) = \vec{a},\ (|\bigcap_1 U|, |\bigcap_2 U|, \ldots, |\bigcap_n U|) = \vec{b} \text{ and } O \succsim U.$$

To see that R^* is well-defined, given that $\succsim$ is a complete preorder, it is sufficient to show that two profiles $V, W \in L^n$ with the same associated vector $\vec{c}$ are indifferent. This conclusion follows easily from Lemmas 5.8 and 5.9.

Transitivity and Completeness of R^* are obvious from the transitivity and completeness of $\succsim$ and the fact that every element in Q^* can be associated with some profile in L^n. □

We have shown in Lemma 5.6 that if a ranking satisfies ANON^n, ASM^n and WISQ2^n, then we can focus exclusively on (Q, R). In Lemma 5.10, we have also shown that a ranking satisfying ANON^n, ASM^n, WISQ^n, AIN^n and CT^n can be interpreted in terms of (Q^*, R^*). We are going to define, similarly to the case of two agents, some properties on these domains.

(1): $(\vec{a}+\vec{1})\,R\,\vec{a}$ for all $\vec{a}\in Q$.

(2): $(a_1,\dots,a_i,\dots,a_j+1,\dots,a_n)\,R\,(a_1,\dots,a_i+1,\dots,a_j,\dots,a_n)$ for all $\vec{a}\in Q$ and $i<j$.

(3): $(a_1,\dots,a_{n-1},a_n+1)\,P\,\vec{a}$ for all $\vec{a}\in Q$ such that $a_{n-1}\neq a_n$.

(4): For all $\vec{c}\in\mathbb{Z}^n$ and all $\vec{a},\vec{b},(\vec{a}+\vec{c}),(\vec{b}+\vec{c})\in Q$,

$$\vec{a}\,R\,\vec{b}\Rightarrow(\vec{a}+\vec{c})\,R\,(\vec{b}+\vec{c}).$$

(1*): $(\vec{a}+\vec{1})\,R^*\,\vec{a}$ for all $\vec{a}\in Q^*$.

(2*): $(\vec{a}+\vec{1})\,R^*\,(\vec{a}+\vec{k})$, for all $\vec{a}\in Q^*$, $\vec{k}\in(\mathbb{N}^*)^n$ such that $(\vec{a}+\vec{k})\in Q^*$, $\sum_{i=1}^{t}k_i\geq t$ for all $t\in I$ and $\sum_{i=1}^{n}k_i=n$.

(3*): $(a_1,\dots,a_{n-1},a_n+1)\,P^*\,\vec{a}$, for all $\vec{a}\in Q^*$ such that $a_n\neq 0$ and $a_{n-1}\neq a_n$.

(4*): For all $\vec{c}\in\mathbb{Z}^n$ and all $\vec{a},\vec{b},(\vec{a}+\vec{c}),(\vec{b}+\vec{c})\in Q^*$ such that $a_n\neq 0$ and $b_n\neq 0$,

$$\vec{a}\,R^*\,\vec{b}\Rightarrow(\vec{a}+\vec{c})\,R^*\,(\vec{b}+\vec{c}).$$

As in the two-agent case, given Lemma 5.6, we have that the axioms have the following implications on (Q,R): ASM^n implies (1), MON^n implies (2) and (3), and WISQ2^n implies (4).

In the same way, given Lemma 5.10, we have that some axioms of the theorem have some implications on (Q^*,R^*). They are the following: ASM^n implies (1*) and (2*), MON^n implies (3*) and WISQ^n implies (4*).

The following claims will be crucial for the rest of the proofs.

Claim 5.8 *If R satisfies (4), then for all $\vec{c}\in\mathbb{Z}^n$ and all $\vec{a},\vec{b},(\vec{a}+\vec{c}),(\vec{b}+\vec{c})\in Q$,*

$$\vec{a}\,R\,\vec{b}\Leftrightarrow(\vec{a}+\vec{c})\,R\,(\vec{b}+\vec{c}).$$

Proof: Using the definition of (4), this claim is trivial. □

Claim 5.9 *If R satisfies (1), (2), (3) and (4) and $i \in (I \setminus \{n\})$, then for all $\vec{x}, \vec{y} \in Q$ such that $x_j = y_j$ for all $j \in (I \setminus \{i, n\})$, we have that there exists $\alpha_i \in \mathbb{R}$ such that*

$$x_n + \alpha_i \cdot x_i > y_n + \alpha_i \cdot y_i \Rightarrow \vec{x} P \vec{y}$$

Proof: Properties (3) and (4) for the cases in which all of the elements of the vectors are equal except two, one of them being the last component (the case proposed in the claim), are equivalent to conditions (3) and (4) of the case of two agents. Then, we can apply Claim 5.7 and we have that there exists $\alpha_i \in (\mathbb{R} \cup \{-\infty, +\infty\})$ such that the inequality in the statement of our claim is true. Now, by (2) we have that $\alpha_i \leq 1$ and, therefore, $\alpha_i \neq +\infty$. We also know from (1) that $(\vec{x} + \vec{1})\, R\, \vec{x}$. Given (2), we can deduce that $(x_1, \ldots, x_i + 1, \ldots, x_n + n - 1)\, R\, \vec{x}$. Therefore, $\alpha_i \neq -\infty$. □

On the basis of the previous claim, we can obtain, for all $i \in (I \setminus \{n\})$ a real value α_i. We will show that these values constitute the weights to which part 4 of our theorem refers. Let two vectors $\vec{x}, \vec{y} \in Q$ be such that $x_n + \sum_{i=1}^{n-1} \alpha_i x_i > y_n + \sum_{i=1}^{n-1} \alpha_i y_i$. We need to prove that $\vec{x} P \vec{y}$. To do so, we divide the proof into the following cases:

1. $\vec{x}$ and $\vec{y}$ have at most one difference in the first $(n-1)$ components. That is, $x_j = y_j$ for all $j \in (I \setminus \{i, n\})$. Then, Claim 5.9 constitutes the proof.[10]

2. $\vec{x}$ and $\vec{y}$ have only two different components in the $(n-1)$ first values of the vectors, $i, j \in \{1, \ldots, n-1\}$. Then, we select $\vec{k} \in Q$ such that $k_i = 2^{(n-i)} \cdot k$, with $k \in \mathbb{N}$. Let $\vec{x}', \vec{y}' \in Q$ such that $\vec{x}' = \vec{x} + \vec{k}$ and $\vec{y}' = \vec{y} + \vec{k}$. Then, by (4) we know that $\vec{x}\, R\, \vec{y} \Leftrightarrow \vec{x}'\, R\, \vec{y}'$. Now, consider the following sets:

[10] We obviously need to add to both parts of the inequality the same (finite) value obtained computing the weighted sum of the other components. These partial sums are equal in both vectors.

$$L(\vec{x}, \vec{y}, \vec{k}) = \{\vec{a^{(t)}} \in Q \text{ with } t \in \mathbb{N}^* \text{ such that } \vec{a^{(t)}} = \vec{x}' + t(\vec{x}' - \vec{y}')\}.$$

$$M(\vec{x}, \vec{y}, \vec{k}) = \{\vec{b^{(t)}} \in Q \text{ with } t \in \mathbb{N}^* \text{ such that } b_l^{(t)} = y_l' \text{ for all } l \in (I \setminus \{i, n\}),\ b_i^{(t)} = a_i^{(t)} \text{ and } b_n^{(t)} = \min\{w_n \in \mathbb{N} \text{ subject to } w_n + \alpha_i b_i^{(t)} > y_n' + \alpha_i y_i'\}\}.$$

It is easy to see that when for some $t_1 \in \mathbb{N}^*$, $\vec{a^{(t_1)}} \in Q$, then $\vec{a^{(t)}} \in Q$ for all $t \in \mathbb{N}^*$ such that $t < t_1$. In the same way, if $\vec{b^{(t_2)}} \in Q$ for some $t_2 \in \mathbb{N}$, then $\vec{b^{(t)}} \in Q$ for all $t \in \mathbb{N}^*$ such that $t < t_2$.

Note that, selecting the appropriate $k \in \mathbb{N}$, we can increase the cardinality of $L(\vec{x}, \vec{y}, \vec{k})$ and $M(\vec{x}, \vec{y}, \vec{k})$ as we need.

Noting that $\vec{a^{(t)}} = \vec{x}' + t(\vec{x}' - \vec{y}')$ and $\vec{a^{(t-1)}} = \vec{y}' + t(\vec{x}' - \vec{y}')$, then by Claim 5.8 we have that $\vec{x}' \, R \, \vec{y}' \Leftrightarrow \vec{a^{(t)}} \, R \, \vec{a^{(t-1)}}$ for all $t \in \mathbb{N}$. Because R is a complete preorder, we have that $\vec{x}' \, R \, \vec{y}' \Leftrightarrow \vec{a^{(t)}} \, R \, \vec{y}'$ for all $t \in \mathbb{N}^*$.

We have that for all $\vec{b^{(t)}} \in M(\vec{x}, \vec{y}, \vec{k})$, $\vec{b^{(t)}}$ and $\vec{y}'$ have only one of the first $(n-1)$ components different (component i). Therefore, we can apply Case 1 and we have that $\vec{b^{(t)}} \, P \, \vec{y}'$. Note that each $\vec{a^{(t)}} \in L(\vec{x}, \vec{y}, \vec{k})$ and its corresponding $\vec{b^{(t)}} \in M(\vec{x}, \vec{y}, \vec{k})$ only differs in one of the first $(n-1)$ components (component j). By definition, we know that

$$(a_n^{(t)} + \textstyle\sum_{i=1}^{n-1} \alpha_i \cdot a_i^{(t)}) - (y_n' + \sum_{i=1}^{n-1} \alpha_i \cdot y_i') = \\ t \cdot \left[(x_n' + \textstyle\sum_{i=1}^{n-1} \alpha_i \cdot x_i') - (y_n' + \sum_{i=1}^{n-1} \alpha_i \cdot y_i')\right].$$

Then, this difference between the weighted sums of $\vec{a^{(t)}}$ and $\vec{y}'$ is increasing with t and there is no limit in the value of this difference. However, we have that there exists $t' \in \mathbb{N}^*$ such that for all $t \in \mathbb{N}$ such that $t > t'$,

$$(b_n^{(t)} + \sum_{i=1}^{n-1} \alpha_i \cdot b_i^{(t)}) - (y_n' + \sum_{i=1}^{n-1} \alpha_i \cdot y_i') \leq 1.^{11}$$

[11]Remember that the existence of t' is guaranteed by selecting a sufficiently high $k \in \mathbb{N}$.

Then, there exist values of $t \in \mathbb{N}$ such that

$$a_n^{(t)} + \sum_{i=1}^{n-1} \alpha_i \cdot a_i^{(t)} > b_n^{(t)} + \sum_{i=1}^{n-1} \alpha_i \cdot b_i^{(t)}.$$

Therefore, for these values of t, $\vec{a^{(t)}}\, P\, \vec{b^{(t)}}$, and, by transitivity, $\vec{a^{(t)}}\, P\, \vec{y}'$.

Then, given that there exists $t \in \mathbb{N}$ such that $\vec{a^{(t)}}\, P\, \vec{y}'$, we have that $\vec{x}'\, P\, \vec{y}'$. Thus, $\vec{x}\, P\, \vec{y}$.

3. The proof in the cases in which there are more than two differences in the $(n-1)$ first components of the vectors follows a similar iterative procedure as in the case of two differences.

We thus have that $x_n + \sum_{i=1}^{n-1} \alpha_i x_i > y_n + \sum_{i=1}^{n-1} \alpha_i y_i \Rightarrow \vec{x} P \vec{y}$. Finally, we also know by (1) that $\sum_{i=1}^{n-1} \alpha_i \geq -1$, and by (2) that $\alpha_i \leq \alpha_{i+1} \leq 1$ for all $i \in \{1, \ldots, n-2\}$. Therefore, $\vec{\alpha} \in D$. The application of Lemma 5.6 concludes the proof of part 4.

As in the two-agent case, parts 1 and 3 clearly follow from the inspection of the rankings satisfying the extra conditions. Similarly, part 2 can be derived by analogy to part 2, as in the two-agent case.

References

[1] Bossert W, Fleuerbaey M, Van de Gaer D (1999) Responsibility, talent, and compensation: A second-best analysis. Rev Econ Design 4: 35–55.

[2] Esteban JM, Ray D (1994) On the measurement of polarization. Econometrica 62: 819–851.

[3] Herrero C (1997) Equitable opportunities: An extension. Econ Letters 55: 91–95.

[4] Herrero C, Iturbe-Ormaetxe I, Nieto J (1998) Ranking opportunity profiles on the basis of the common opportunities. Math Soc Sci 35: 273–289.

[5] Kranich L (1996) Equitable opportunities: An axiomatic approach. J Econ Theory 71: 131–147.

[6] Ok EA (1997) On opportunity inequality measurement. J Econ Theory 77: 300–329.

[7] Ok EA, Kranich L (1998) The measurement of opportunity inequality: A cardinality-based approach. Soc Choice Welfare 15: 263–287.

[8] Pattanaik PK, Xu Y (1990) On ranking opportunity sets in terms of freedom of choice. Rech Econ Louvain 56: 383–390.

[9] Peragine V (1999) The distribution and the redistribution of opportunities. J Econ Surveys 13: 37–69.

[10] Rawls J (1971) A Theory of Justice. Harvard University Press, Cambridge, MA.

[11] Savaglio E, Vanucci S. Filtral preorders and opportunity inequality, J Econ Theory, forthcoming.

[12] Sen AK (1985) Commodities and Capabilities. North-Holland, Amsterdam.

[13] Weymark JA (2003) Generalized Gini indices of equality of opportunity. J Econ Inequality 1: 5–24.