

Indivisible Commodities and Decentralization of Strong Core Allocations

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Abstract

We consider a pure exchange economy with finitely many indivisible commodities that are available only in integer quantities. We show that in such an economy with sufficiently large agents, but finitely many agents, every strong core allocation, if it exists, can be supported as a Walras equilibrium.

Keywords: Indivisible commodities, Strong core, Walras equilibrium.

JEL Classification: C71.

1 Introduction

The core is an institution-free concept, but it is known that in an economy with perfectly divisible commodities, core allocations can be approximately decentralized by prices as the number of participants becomes large. The purpose of this paper is to show that if every commodity is indivisible, then core allocations can be *exactly* decentralized by prices, even though the number of economic agents is finite.

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The perfect divisibility of commodities is usually assumed in economic theory for the convenience of the analysis. In this paper, we assume that every commodity can be consumed only in integer quantities. Agents can consume multiple types of commodities, and can consume multiple units of each commodity. Thus, the commodity space is given by the products of the set of integers. Our argument shows that the inherent properties of the set of integers, such as countability, discreteness, and additivity, are helpful, rather than obstructive, to prove the decentralization of core allocations.

In the literature, several notions of improvement by a coalition are used to define cores. In an economy with indivisible commodities, the size of the core depends heavily on which notion of improvement is adopted. Accordingly, a clear distinction among the several competing notions of cores should be made. Debreu and Scarf [4] defined a core by weak improvement that requires some members in a coalition to be better off, and other members not to be worse off, by redistribution of their endowments. We refer to this notion of the core as the *strong core*.¹

Debreu and Scarf [4] considered a sequence of replica economies with convex consumption sets. Two agents who have the same preference relation and the same endowment vector are said to be of the same type. An economy where in each type there are n times as many agents as the original economy is called the n -fold replica economy. If agents' preference relations are strongly convex, then agents of the same type are allocated the same consumption bundle by a strong core allocation or a Walras allocation.² Thus, by choosing a representative agent from each type, we can regard strong core allocations and Walras allocations for any replica economy as elements of a Euclidean space with the same dimension as that of the original economy. Debreu and Scarf [4] showed that under the assumption of local nonsatiation of agents' preference relations, every Walras allocation is in the strong core. The more replications of an economy, the more possible coalitions. Hence, the sequence of strong cores is shrinking, whereas the sequence of Walras allocations is constant. Under the assumptions of strong convexity and local nonsatiation of preference relations, Debreu and Scarf [4] proved that the limit of a decreasing sequence of strong cores coincides with the set of Walras allocations.

¹We use the same terminology as Inoue [8] concerning the several notions of cores.

²We refer to this property of strong core allocations as the *strong equal treatment* property.

Anderson [1] used the notion of a core defined by strong improvement that requires all members in a coalition to be better off by redistribution of their endowments. We refer to this as the *weak core*. Without any assumptions, Walras allocations are always in the weak core. Anderson [1] considered a more general sequence of economies than Debreu and Scarf [4]. In Anderson’s model, all agents may belong to different types, and agents’ preference relations need not be convex. Anderson [1] proved that under the assumption of monotonicity of preference relations (a stronger assumption than local nonsatiation), if the number of agents whose endowment vectors are in a given bounded set increases, then some measure of the non-Walras degree of weak core allocations tends to zero. Therefore, in a large finite economy, weak core allocations can be approximately decentralized by prices.

In both Debreu and Scarf [4] and Anderson [1], the assumptions of the local nonsatiation of preference relations and the convexity of consumption sets play essential roles. Although our economy does not have these properties, we can show that if the number of agent types is finite, and in each type there are many agents, then each strong core allocation is a Walras allocation.

In our economy, in contrast to Debreu and Scarf [4], there can exist a Walras allocation that is not in the strong core because of the local satiation of preference relations. More particularly, in our economy, the coincidence between the strong core and the set of Walras allocations is not obtained in general. In addition, once again in contrast to Debreu and Scarf [4], our theorem is not a “limit theorem” since in our economy, finitely many agents are required for strong core allocations to be decentralized by prices. We give a type set first and require each type to have a sufficiently large number of agents. Hence, our theorem has the flavor of a so-called “type sequence,” which is more general than a sequence of replica economies but more restrictive than that investigated by Anderson [1].

It should be noted that in our economy, however, even a Walras equilibrium may not exist because of indivisibility.³ In such a case, if the number of agents is large, the strong core is also empty. We give an example (Example 1 in Section 2) of an economy where the

³A sufficient condition for the existence of a Walras equilibrium has not known yet although a sufficient condition for the nonemptiness of the weak core is given by Inoue [9].

strong core is nonempty and is a strictly smaller set than the set of Walras allocations. In addition, we give an example (Example 2 in Section 2) of an economy where the strong core is empty but a Walras equilibrium exists. Therefore, in Examples 1 and 2, the strong core is strictly smaller than the set of Walras allocations. In Example 3 in Section 2, we give an economy where the strong core coincides with the set of Walras allocations. Hence, the strong core is not always strictly smaller than the set of Walras allocations.

In the process of the proof of our main theorem, the weak equal treatment property of strong core allocations is shown. That is, consumption bundles allocated by a strong core allocation to agents of the same type have the same utility level with respect to the common preference relation (see Lemma 2 in Section 3). As discussed earlier, in an economy with convex consumption sets and strongly convex preference relations, strong core allocations have the strong equal treatment property. (All members of the same type receive the same consumption bundle.) This property depends heavily on the strong convexity of preference relations. In addition, as Green [6] pointed out, this property is inherent to replica economies. Green [6] proved that for almost all economies where the greatest common divisor of the numbers of agents of each type is one, there exists a strong core allocation that does not even have the weak equal treatment property. On the other hand, the weak equal treatment property of our economy is related to the fact that the number of resource-feasible allocations within a coalition is bounded by a constant that is independent of the number of agents because of the discreteness of the commodity space. It should be emphasized that the method of proof of our equal treatment property is very different from that provided by Debreu and Scarf [4]. Moreover, our equal treatment property holds even if the greatest common divisor of the numbers of agents of each type is one.

In other work, Inoue [8] considered an atomless economy with the same commodity space as ours. He introduced a core defined by improvement as an intermediate notion between the weak and the strong improvement. Accordingly, such a core is also an intermediate concept between the strong and the weak core. We denote this as the *core*. Inoue [8] showed that under some assumptions in an atomless economy, the core coincides with the set of exactly feasible Walras allocations. Despite the fact that our assumptions on agents' preference relations are stronger than Inoue's [8], in a finite economy with

a sufficiently large number of agents, we can just show that strong core allocations are Walras allocations. As was shown by Inoue [10], in a finite economy there can exist a core allocation that is not a Walras allocation, even if the number of agents is arbitrarily large.

Shapley and Scarf [12] analyzed yet another type of indivisible commodity market. Their economic model has finitely many agents, and each agent has only one indivisible commodity (e.g., a house). Commodities are also differentiated; therefore, the number of agents is equal to the number of commodities. In their model, it is assumed that every agent prefers his commodity to nothing. Hence, a resource-feasible allocation that satisfies individual rationality can be represented by a permutation of the initial allocation. By using David Gale's top-trading-cycle method, Shapley and Scarf proved that Walras equilibria always exist, even though the strong core can be empty. Subsequently, Wako [13] showed that in the Shapley-Scarf model, each strong core allocation is a Walras allocation. Wako's [13] proof depends heavily on the model specification, so it is very different from the proof in this paper.

The paper itself is organized as follows. In Section 2, we present our model and the main theorem. Section 3 provides the proof of the theorem. Purely technical results used in the proof of the main theorem are relegated to the appendix.

2 The Model and the Main Theorem

We begin with some notation. Let \mathbb{R} , \mathbb{Q} , and \mathbb{Z} be the sets of real numbers, rational numbers, and integers, respectively. For $x = (x^{(1)}, \dots, x^{(m)})$ and $y = (y^{(1)}, \dots, y^{(m)})$ in \mathbb{R}^m ($m \geq 2$), we write $x \geq y$ if $x^{(j)} \geq y^{(j)}$ for all $j \in \{1, \dots, m\}$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x^{(j)} > y^{(j)}$ for all $j \in \{1, \dots, m\}$. The symbol 0 denotes the origin in \mathbb{R}^m , as well as the real number zero. Let χ_i be the i th unit vector, i.e., $\chi_i^{(i)} = 1$ and $\chi_i^{(j)} = 0$ if $j \neq i$. $\mathbb{R}_+^m = \{x \in \mathbb{R}^m \mid x \geq 0\}$; $\mathbb{R}_{++}^m = \{x \in \mathbb{R}^m \mid x \gg 0\}$. \mathbb{Q}_+^m , \mathbb{Q}_{++}^m , \mathbb{Z}_+^m , and \mathbb{Z}_{++}^m are defined in a similar way. Note that \mathbb{Z}_{++} is the set of natural numbers. The inner product $\sum_{j=1}^m x^{(j)}y^{(j)}$ of x and y in \mathbb{R}^m is denoted by $x \cdot y$. For x in \mathbb{R}^m , let $\|x\|_1 = \sum_{j=1}^m |x^{(j)}|$ and $\|x\|_\infty = \max\{|x^{(j)}| \mid j = 1, \dots, m\}$. The cardinality of a finite set A is denoted by $\#A$. For a subset C of \mathbb{R}^m , denote by $\text{co}(C)$ the convex hull of C .

We consider a pure exchange economy with L indivisible commodities, where L is a natural number with $L \geq 2$.⁴ Every commodity in our economy is available in integer quantities. Therefore, the commodity space is given by \mathbb{Z}^L . For simplicity, we assume that agents have the same consumption set \mathbb{Z}_+^L . An agent a of our economy is characterized by preference relation \succsim_a on \mathbb{Z}_+^L and endowment vector $e(a) \in \mathbb{Z}_+^L$. A preference relation \succsim on \mathbb{Z}_+^L is required to have the following two properties. First, \succsim is a reflexive, transitive, and complete binary relation on \mathbb{Z}_+^L . The second requirement is that \succsim is weakly monotone, i.e., for all x and y in \mathbb{Z}_+^L , if $x \leq y$, then $x \succsim y$.⁵ Let \mathcal{P} be the set of all preference relations on \mathbb{Z}_+^L . Hence, the space of agents' characteristics is $\mathcal{P} \times \mathbb{Z}_+^L$. A mapping \mathcal{E} of a finite set A of agents into $\mathcal{P} \times \mathbb{Z}_+^L$, $\mathcal{E}(a) = (\succsim_a, e(a))$ for all $a \in A$, is an *economy* if $\sum_{a \in A} e(a) \gg 0$. Given an economy $\mathcal{E} : A \rightarrow \mathcal{P} \times \mathbb{Z}_+^L$, an *allocation* for \mathcal{E} is a mapping of A into \mathbb{Z}_+^L . An allocation $f : A \rightarrow \mathbb{Z}_+^L$ for \mathcal{E} is *exactly feasible* if the equality $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$ holds. A *coalition* is a nonempty subset of A .

We give the precise definition of the strong core.

Definition 1 Let $f : A \rightarrow \mathbb{Z}_+^L$ be an allocation for an economy $\mathcal{E} : A \rightarrow \mathcal{P} \times \mathbb{Z}_+^L$. A coalition S can *weakly improve upon* f if there exists a mapping $g : S \rightarrow \mathbb{Z}_+^L$ such that

$$\begin{aligned} \sum_{a \in S} g(a) &= \sum_{a \in S} e(a), \\ g(a) &\succ_a f(a) \quad \text{for some } a \in S, \text{ and} \\ g(a) &\succsim_a f(a) \quad \text{for all } a \in S. \end{aligned}$$

The set of all exactly feasible allocations for \mathcal{E} that cannot be weakly improved upon by any coalition is called the *strong core* of \mathcal{E} and is denoted by $C_S(\mathcal{E})$.

Obviously, every strong core allocation is then Pareto-efficient. If at least one agent has a *strongly monotone* preference relation \succ , i.e., for all x and y in \mathbb{Z}_+^L , $x > y$ implies $x \succ y$, then the size of the strong core does not change even if we replace the exact feasibility

⁴For an economy with only one commodity, we can easily show that if every agent's preference relation is weakly monotone and if at least two agents' preference relations are strongly monotone, then every strong core allocation is a Walras allocation.

⁵For a preference relation \succsim , we define binary relations \succ and \sim as usual: $x \succ y$ if and only if not $(x \precsim y)$; $x \sim y$ if and only if $x \precsim y$ and $y \precsim x$. We sometimes write $x \succsim y$ for $y \precsim x$ and write $x \prec y$ for $y \succ x$.

constraint in Definition 1 by a weak feasibility constraint $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$ and $\sum_{a \in S} g(a) \leq \sum_{a \in S} e(a)$. It should be noted that the strong core can be empty by the indivisibility of the commodities (see Example 2 below).

We next provide the definition of a Walras equilibrium.

Definition 2 Let $\mathcal{E} : A \rightarrow \mathcal{P} \times \mathbb{Z}_+^L$ be an economy. A pair (p, f) of a price vector $p \in \mathbb{Q}_+^L$ and an exactly feasible allocation $f : A \rightarrow \mathbb{Z}_+^L$ is called a *Walras equilibrium* for \mathcal{E} if

- (i) for all $a \in A$, $p \cdot f(a) \leq p \cdot e(a)$; and
- (ii) for all $a \in A$, if $x \in \mathbb{Z}_+^L$ and $x \succ_a f(a)$, then $p \cdot x > p \cdot e(a)$.

An exactly feasible allocation $f : A \rightarrow \mathbb{Z}_+^L$ is called a *Walras allocation* for \mathcal{E} if there exists a price vector $p \in \mathbb{Q}_+^L$ such that (p, f) is a Walras equilibrium for \mathcal{E} . The set of all Walras allocations for \mathcal{E} is denoted by $W^*(\mathcal{E})$.

If (p, f) is a Walras equilibrium, then for all $\alpha \in \mathbb{Q}_{++}$, $(\alpha p, f)$ is also a Walras equilibrium. Thus, for every Walras allocation, there exists an associated equilibrium price vector that is an element of \mathbb{Z}_+^L . It should be noted that we can restrict the space of price vectors to \mathbb{Q}_+^L without loss of generality. In fact, if a pair of a vector $p \in \mathbb{R}_+^L \setminus \mathbb{Q}_+^L$ and a feasible allocation f satisfies conditions (i) and (ii) of Definition 2, then there exists a price vector $p' \in \mathbb{Q}_+^L$ such that (p', f) is a Walras equilibrium.⁶

In contrast to the size of the strong core, if we replace the exact feasibility constraint in Definition 2 by a weak feasibility constraint $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$, then the size of the set of Walras allocations can be larger. Namely, there can exist a pair (p, f) of a price vector p and an allocation f such that $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$ and conditions (i) and (ii) from Definition 2 are satisfied.⁷

Note that our economy may not have a Walras equilibrium.⁸ Note also that Walras allocations, if they exist, are not always Pareto-efficient, although Walras allocations

⁶This depends on the fact that in a finite economy, the endowment mapping $e : A \rightarrow \mathbb{Z}_+^L$ is always bounded. The existence of such a price vector p' follows from Remark 5.3 of Inoue [8].

⁷See Inoue [8, Example 2.5].

⁸Henry [7] gave an example of an economy with one indivisible commodity and two divisible commodities such that a Walras equilibrium does not exist. For economies where every commodity is indivisible, Shapley and Scarf [12, Section 8] gave an example of the nonexistence of a Walras equilibrium.

must be weakly Pareto-efficient. Therefore, Walras allocations are not always strong core allocations. This is because every agent's preference relation is locally satiated. In Example 1 below, we give an economy \mathcal{E} such that $\emptyset \neq C_S(\mathcal{E}) \subsetneq W^*(\mathcal{E})$.

We place restrictions on preference relations. For all $k \geq 2$,⁹ we define a subset \mathcal{P}_k of \mathcal{P} as follows: $\succsim \in \mathcal{P}_k$ if and only if (i) $\succsim \in \mathcal{P}$ and (ii) for all $h, i \in \{1, \dots, L\}$ with $h \neq i$ and all $x \in \mathbb{Z}_+^L$, if $x^{(i)} \geq 1$, then $x + k\chi_h - \chi_i \succ x$.¹⁰ From (i) and (ii), it follows that for all $h \in \{1, \dots, L\}$ and all $x \in \mathbb{Z}_+^L$, $x + k\chi_h \succ x$ holds.¹¹ Condition (ii) means that agents whose preference relations are in \mathcal{P}_k are willing to give up one unit of a commodity in exchange for k units of another commodity. Therefore, preference relations in \mathcal{P}_k have uniformly positive marginal rates of substitution. In particular, the lexicographic ordering is excluded.

In the theorem below, we first give a natural number $k \geq 2$, a nonempty finite subset T of $\mathcal{P}_k \times \mathbb{Z}_+^L$, and a real number $r \geq \#T$. This set T is called a type set of agents. Then, we consider an economy where the ratio of agents of each type to the whole economy is greater than or equal to $1/r$ and the total number of agents is sufficiently large. Therefore, the number of agents of each type is also large. For all $t \in T$, we write $t = (\succsim_t, e_t)$. Given an economy $\mathcal{E} : A \rightarrow T$ and a type $t \in T$, denote the set of agents of type t by A_t , i.e., $A_t = \mathcal{E}^{-1}(\{t\})$.

Now, we are ready to state the main result.

Theorem 1 *For all $r \in \mathbb{R}$ with $r \geq 1$, all $k \in \mathbb{Z}$ with $k \geq 2$, and all $T \subset \mathcal{P}_k \times \mathbb{Z}_+^L$ with $\#T \leq r$ and $\sum_{t \in T} e_t \gg 0$, there exists an $N \in \mathbb{Z}_{++}$ such that if $\mathcal{E} : A \rightarrow T$, $\#A \geq N$, and $\#A_t/\#A \geq 1/r$ for all $t \in T$, then $C_S(\mathcal{E}) \subset W^*(\mathcal{E})$.*

The number N depends on r , k , L , and $M = \max\{\|e\|_\infty \mid (\succsim, e) \in T\}$. Note that an

⁹The reason why k is assumed to be greater than 1 is that \mathcal{P}_1 , which is defined similar to \mathcal{P}_k with $k \geq 2$, is empty. Indeed, if $\succsim \in \mathcal{P}_1$, then $\chi_1 = \chi_2 + \chi_1 - \chi_2 \succ \chi_2$ and $\chi_2 = \chi_1 + \chi_2 - \chi_1 \succ \chi_1$, contradicting the irreflexivity of \succ . This fact was pointed out by Akiyoshi Shioura.

¹⁰Condition (ii) is related to the equi-monotonicity of preference relations in an economy with divisible commodities. Let \mathcal{Q} be the space of continuous and strongly monotone preference relations on the consumption set \mathbb{R}_+^L . It can be shown that for every finite subset \mathcal{Q}' of \mathcal{Q} and every compact subset K of \mathbb{R}_+^L , there exists a positive number δ such that for all $\succsim \in \mathcal{Q}'$, all $h, i \in \{1, \dots, L\}$, and all $x \in K$, $x + \chi_h - \delta\chi_i \succ x$ holds.

¹¹This fact can be shown as follows. Let $i \neq h$. Then, $x + k\chi_h = (x + \chi_i) + k\chi_h - \chi_i \succ x + \chi_i \succsim x$.

economy $\mathcal{E} : A \rightarrow T$ with $\#T = r$ and $\#A_t/\#A \geq 1/r$ for all $t \in T$ is the $\#A/r$ -fold replica economy. The proof of the theorem is given in the next section.

Our theorem says that the inclusion $C_S(\mathcal{E}) \subset W^*(\mathcal{E})$ holds for a large finite economy. An economy $\mathcal{E}_1 : A \rightarrow \mathcal{P}_k \times \mathbb{Z}_+^L$ can have a strong core allocation $f_1 : A \rightarrow \mathbb{Z}_+^L$ that is not a Walras allocation. Then, for the n -fold replica economy $\mathcal{E}_n : A \times \{1, \dots, n\} \rightarrow \mathcal{P}_k \times \mathbb{Z}_+^L$ of \mathcal{E}_1 defined by $\mathcal{E}_n(a, i) = \mathcal{E}_1(a)$ for all $(a, i) \in A \times \{1, \dots, n\}$, a symmetric replica allocation $f_n : A \times \{1, \dots, n\} \rightarrow \mathbb{Z}_+^L$ of f_1 defined by $f_n(a, i) = f_1(a)$ for all $(a, i) \in A \times \{1, \dots, n\}$ is not a Walras allocation for \mathcal{E}_n as before, and if n is sufficiently large, from our theorem, it follows that f_n is not a strong core allocation for \mathcal{E}_n . The next example illustrates this fact.

Example 1 Let $L = 2$ and $A = \{a, b\}$. Each agent's endowment vector is given by $e(a) = (3, 1)$ and $e(b) = (1, 3)$. Agents' preference relations are represented by the following utility functions (see Figures 1 and 2):

$$\begin{aligned} u_a(x^{(1)}, x^{(2)}) &= \begin{cases} 2x^{(1)} + x^{(2)} & \text{if } x^{(1)} \leq 1, \\ \frac{1}{2}(x^{(1)} + 2x^{(2)} + 3) & \text{if } x^{(1)} \geq 2, \end{cases} \quad \text{and} \\ u_b(x^{(1)}, x^{(2)}) &= x^{(1)} + x^{(2)}. \end{aligned}$$

Clearly, both \succsim_a and \succsim_b are in \mathcal{P}_3 . We call this economy \mathcal{E}_1 .¹²

An allocation $f_1 : A \rightarrow \mathbb{Z}_+^2$ for \mathcal{E}_1 is defined by $f_1(a) = (1, 2)$ and $f_1(b) = (3, 2)$. One could easily check that $f_1 \in C_S(\mathcal{E}_1)$ and $f_1 \notin W^*(\mathcal{E}_1)$.

For all $n \geq 1$, let $\mathcal{E}_n : A \times \{1, \dots, n\} \rightarrow \mathcal{P}_3 \times \mathbb{Z}_+^2$ be the n -fold replica economy of \mathcal{E}_1 , i.e., $\mathcal{E}_n(c, i) = \mathcal{E}_1(c)$ for all $(c, i) \in A \times \{1, \dots, n\}$. For all $n \geq 1$, let $e_n : A \times \{1, \dots, n\} \rightarrow \mathbb{Z}_+^2$ be the endowment mapping of economy \mathcal{E}_n , and let $f_n : A \times \{1, \dots, n\} \rightarrow \mathbb{Z}_+^2$ be the symmetric replica allocation of f_1 , i.e., $f_n(c, i) = f_1(c)$ for all $(c, i) \in A \times \{1, \dots, n\}$. For all $n \geq 1$, allocation f_n is not a Walras allocation for \mathcal{E}_n for the same reason that $f_1 \notin W^*(\mathcal{E}_1)$. Although allocation f_1 is a strong core allocation for \mathcal{E}_1 , for all $n \geq 2$, its symmetric replica allocation f_n is not a strong core allocation for \mathcal{E}_n as we show in the following.

Let $n \geq 2$. Consider a coalition $S = \{(a, 1), (a, 2), (b, 1)\}$. A mapping $g : S \rightarrow \mathbb{Z}_+^2$ is

¹²This economy is similar to an economy from Example 2.5 of Inoue [8].

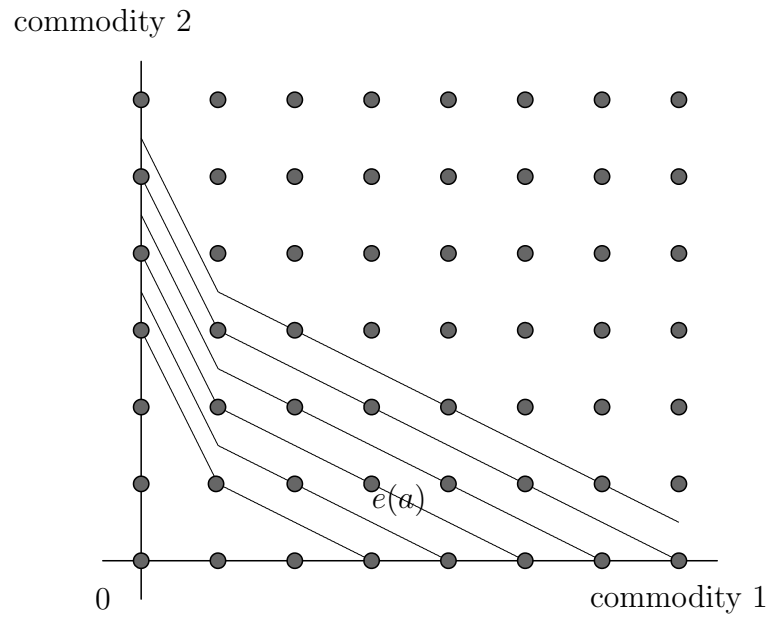


Figure 1: Endowment vector and indifference curves of agent a

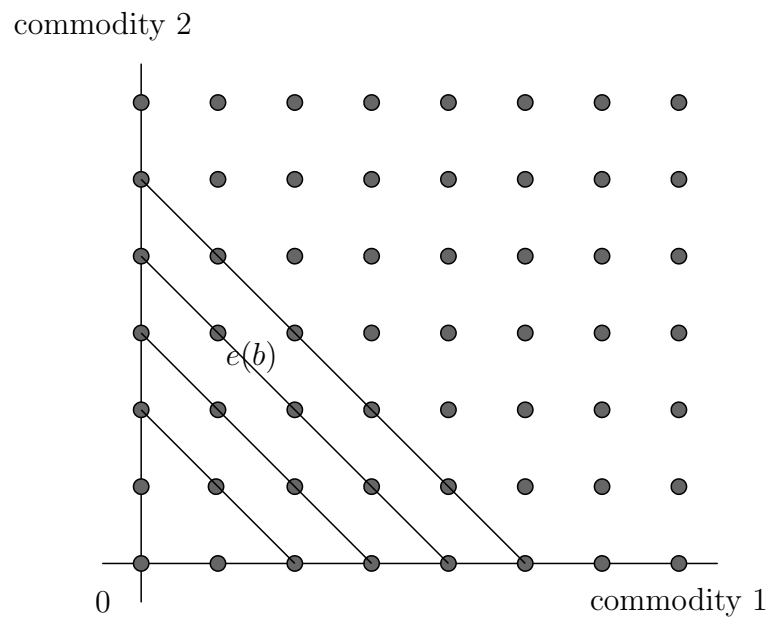


Figure 2: Endowment vector and indifference curves of agent b

defined by

$$\begin{aligned} g(a, i) &= f_1(a) = (1, 2) \quad \text{if } i \in \{1, 2\}, \text{ and} \\ g(b, 1) &= (5, 1). \end{aligned}$$

We can easily obtain that

$$\sum_{(c,i) \in S} g(c, i) = \sum_{(c,i) \in S} e_n(c, i) \quad \text{and} \quad g(b, 1) \succ_b f_n(b, 1),$$

so $f_n \notin C_S(\mathcal{E}_n)$ for all $n \geq 2$.

From our theorem, it follows that there exists an $n_0 \in \mathbb{Z}_{++}$ such that for all $n \geq n_0$, $C_S(\mathcal{E}_n) \subset W^*(\mathcal{E}_n)$ holds. In this example, one can choose 2 as n_0 . It can be shown that for all $n \geq 2$,

$$C_S(\mathcal{E}_n) = \left\{ f : A \times \{1, \dots, n\} \rightarrow \mathbb{Z}_+^2 \left| \begin{array}{l} f(a, i) = (1, 3) \quad \text{for all } i \in \{1, \dots, n\}, \\ \|f(b, i)\|_1 = 4 \quad \text{for all } i \in \{1, \dots, n\}, \text{ and} \\ \sum_{(c,i) \in A \times \{1, \dots, n\}} f(c, i) = \sum_{(c,i) \in A \times \{1, \dots, n\}} e_n(c, i) \end{array} \right. \right\}.$$

Therefore, for all $n \geq 2$, every allocation in $C_S(\mathcal{E}_n)$ is a Walras allocation under the price vector $(1, 1)$. Thus, for all $n \geq 2$, $\emptyset \neq C_S(\mathcal{E}_n) \subset W^*(\mathcal{E}_n)$.

Note that, for all $n \geq 1$, e_n is a Walras allocation under the price vector $p = (1, p^{(2)})$ with $1 < p^{(2)} < 4/3$, but e_n is not a strong core allocation. By summing up these facts, for all $n \geq 2$, $\emptyset \neq C_S(\mathcal{E}_n) \subsetneq W^*(\mathcal{E}_n)$ holds.

The next example says that for some economies the strong core is empty, but the set of Walras allocations is not empty. Therefore, the existence of a Walras allocation does not always imply the nonemptiness of the strong core.

Example 2 Let $L = 2$ and $A = \{a\}$. Agent a 's endowment vector $e(a)$ is given by $(1, 2)$. The preference relation \succsim_a of agent a is represented by a utility function $u_a : \mathbb{Z}_+^2 \rightarrow \mathbb{R}$ defined by

$$u_a(x^{(1)}, x^{(2)}) = \begin{cases} 3.5 & \text{if } (x^{(1)}, x^{(2)}) = (3, 0), \\ x^{(1)} + x^{(2)} & \text{otherwise.} \end{cases}$$

(See Figure 3.) Thus, $\succsim_a \in \mathcal{P}_2$ and \succsim_a is strongly monotone. We call this economy \mathcal{E}_1 . For all $n \geq 1$, let $\mathcal{E}_n : A \times \{1, \dots, n\} \rightarrow \mathcal{P}_2 \times \mathbb{Z}_+^2$ be the n -fold replica economy of \mathcal{E}_1 . Let $e_n : A \times \{1, \dots, n\} \rightarrow \mathbb{Z}_+^2$ be the endowment mapping of economy \mathcal{E}_n .

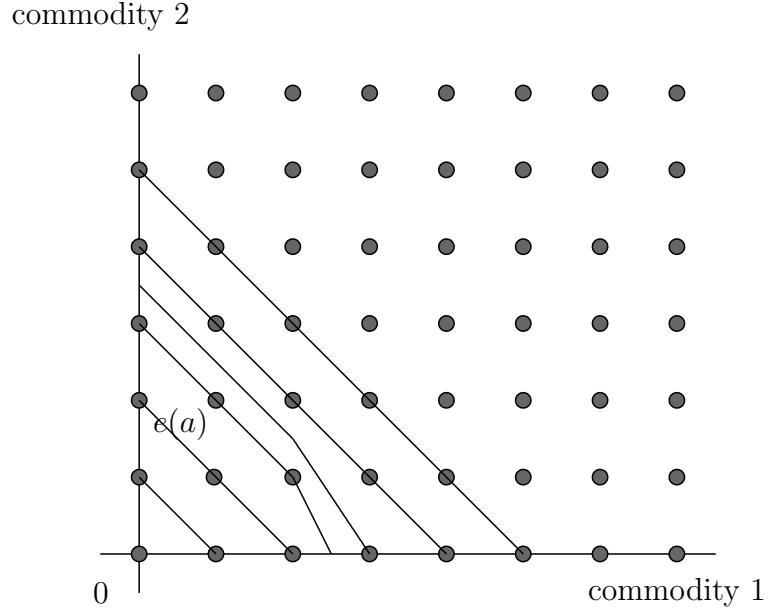


Figure 3: Endowment vector and indifference curves of agent a

Under the price vector $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$, a pair (p, e_n) is a Walras equilibrium of \mathcal{E}_n for all $n \in \mathbb{Z}_{++}$. Hence, $e_n \in W^*(\mathcal{E}_n)$ for all $n \in \mathbb{Z}_{++}$. Note that for all $n \in \mathbb{Z}_{++}$ and all $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$, (p, e_n) is a unique Walras equilibrium. For all $n \in \mathbb{Z}_{++}$, if the price vector $p = (1, p^{(2)})$ satisfies $p^{(2)} \leq 1/2$, then commodity 2 has excess demand. On the other hand, if $p = (1, p^{(2)})$ satisfies $p^{(2)} \geq 1$, then commodity 1 has excess demand. Thus, the price vector $p = (1, p^{(2)})$ with $p^{(2)} \leq 1/2$ or $p^{(2)} \geq 1$ cannot be an equilibrium price vector. Therefore, $\{e_n\} = W^*(\mathcal{E}_n)$ for all $n \in \mathbb{Z}_{++}$.

It can be shown that $e_n \notin C_S(\mathcal{E}_n)$ for all $n \geq 3$. Actually, a coalition with three agents can weakly improve upon e_n . From our theorem, it follows that $C_S(\mathcal{E}_n) \subset W^*(\mathcal{E}_n)$ for sufficiently large n . Therefore, $\emptyset = C_S(\mathcal{E}_n) \subsetneq W^*(\mathcal{E}_n) = \{e_n\}$ for n large enough. Indeed, the above strict inclusion is met for all $n \geq 4$. (One can show that $C_S(\mathcal{E}_n) = \emptyset$ for all $n \geq 4$.) On the other hand, if $2 \leq n \leq 3$, there exists a strong core allocation that is not a Walras allocation. Actually, an allocation $f : A \times \{1, 2\} \rightarrow \mathbb{Z}_+^2$ for \mathcal{E}_2 defined by $f(a, 1) = (0, 3)$ and $f(a, 2) = (1, 2)$ is a strong core allocation but is not a Walras allocation. Also, an allocation $g : A \times \{1, 2, 3\} \rightarrow \mathbb{Z}_+^2$ for \mathcal{E}_3 defined by

$$g(a, 1) = (3, 0), \quad \text{and}$$

$$g(a, i) = (0, 3) \quad \text{for all } i \in \{2, 3\}$$

is a strong core allocation but is not a Walras allocation.

In Examples 1 and 2, the strict inclusion $C_S(\mathcal{E}_n) \subsetneq W^*(\mathcal{E}_n)$ holds for sufficiently large n . The inclusion is not always strict, that is, there can exist a large finite economy \mathcal{E} with the equality $C_S(\mathcal{E}) = W^*(\mathcal{E})$. The next example illustrates this fact.

Example 3 Let $L = 2$ and $A = \{a\}$. Agent a 's endowment vector $e(a)$ is given by $(1, 2)$ and agent a 's preference relation \succsim_a is given by a utility function $u_a(x^{(1)}, x^{(2)}) = x^{(1)} + x^{(2)}$. Note that $\succsim_a \in \mathcal{P}_2$ and \succsim_a is strongly monotone. This economy is different from economy \mathcal{E}_1 from Example 2 only about the utility level at $(3, 0)$. We call this economy \mathcal{E}'_1 .

It can be shown that for all $n \geq 1$,

$$\begin{aligned} C_S(\mathcal{E}'_n) &= \left\{ f : A \times \{1, \dots, n\} \rightarrow \mathbb{Z}_+^2 \left| \begin{array}{l} \|f(a, i)\|_1 = 3 \quad \text{for all } i \in \{1, \dots, n\} \text{ and} \\ \sum_{i=1}^n f(a, i) = n(1, 2) \end{array} \right. \right\} \\ &= W^*(\mathcal{E}'_n), \end{aligned}$$

where \mathcal{E}'_n is the n -fold replica economy of \mathcal{E}'_1 . Therefore, the equality $C_S(\mathcal{E}'_n) = W^*(\mathcal{E}'_n)$ holds for all $n \geq 1$.

3 Proof of Theorem

We state the method of our proof before giving a formal proof. Let $r \geq 1$ and $k \in \mathbb{Z}$ with $k \geq 2$. Let $T \subset \mathcal{P}_k \times \mathbb{Z}_+^L$ be a type set where $\#T \leq r$, $\sum_{t \in T} e_t \gg 0$, and for some $t \in T$, \succsim_t is strongly monotone. Let $\mathcal{E} : A \rightarrow T$ be an economy such that $\#A$ is sufficiently large and for all $t \in T$, $\#A_t / \#A \geq 1/r$. Let $M = \max\{\|e\|_\infty \mid (\succsim, e) \in T\}$.

Let f be a strong core allocation for \mathcal{E} . Let $H^f = \text{span}\{f(a) - e(a) \mid a \in A\}$ and for all $a \in A$, let

$$\varphi^f(a) = \{z \in \mathbb{Z}^L \mid z + e(a) \in \mathbb{Z}_+^L \quad \text{and} \quad z + e(a) \succ_a f(a)\}.$$

Note that the set $\bigcup_{a \in A} \varphi^f(a)$ is bounded below and, by the weak monotonicity of preference relations, $\bigcup_{a \in A} \varphi^f(a) = \bigcup_{a \in A} \varphi^f(a) + \mathbb{Z}_+^L$ holds. Thus, if $\text{co}(\bigcup_{a \in A} \varphi^f(a)) \cap H^f = \emptyset$, then, by Lemma 6 in the appendix, there exists a price vector $p \in H^{f^\perp} \cap \mathbb{Z}_+^L$ and a positive real number ε such that for all $z \in \text{co}(\bigcup_{a \in A} \varphi^f(a))$, $p \cdot z \geq \varepsilon$ holds, where H^{f^\perp} is the orthogonal complement of H^f . From $p \in H^{f^\perp}$, it follows that $p \cdot f(a) = p \cdot e(a)$ for all $a \in A$,

i.e., allocation f satisfies agents' budget constraints. If $a \in A$, $x \in \mathbb{Z}_+^L$, and $x \succ_a f(a)$, then $x - e(a) \in \varphi^f(a)$. Hence, $p \cdot (x - e(a)) \geq \varepsilon > 0$. Therefore, allocation f is a Walras allocation. By this argument, it suffices to show that for every strong core allocation f , $\text{co}(\bigcup_{a \in A} \varphi^f(a)) \cap H^f = \emptyset$ holds.

In Lemma 1, we show that strong core allocations are uniformly bounded, say, $\|f(a)\|_\infty \leq \xi$ for all $f \in C_S(\mathcal{E})$ and all $a \in A$. It should be emphasized that we can make the bound ξ independent of the size $\#A$ of the economy. In Lemma 2, we show that if the number of agents is sufficiently large, all strong core allocations have the weak equal treatment property, i.e., $f(a) \sim_t f(b)$ for all $f \in C_S(\mathcal{E})$, all $t \in T$, and all $a, b \in A_t$. This property enables us to write $\varphi_t^f = \varphi^f(a)$ for all $a \in A_t$. To obtain Lemma 2, we use Lemma 1 and Lemma 3 in the appendix.

By using the fact that the marginal rates of substitution of agents' preference relations are uniformly far away from zero, we can see that for sufficiently large ρ , $\text{co}(\bigcup_{t \in T} \varphi_t^f) \cap H^f = \emptyset$ if and only if $\text{co}(\bigcup_{t \in T} \varphi_t^f \cap X_{L,\rho}) \cap H^f = \emptyset$, where $X_{L,\rho} = \{x \in \mathbb{Z}^L \mid \|x\|_\infty \leq \rho\}$. From the uniform boundedness of strong core allocations (Lemma 1), we can make the number ρ independent of the strong core allocation f and dependent only on exogenous variables r, k, L , and M . Hence, our final goal is to show that for every strong core allocation f , $\text{co}(\bigcup_{t \in T} \varphi_t^f \cap X_{L,\rho}) \cap H^f = \emptyset$. In other words, we should find a coalition that can weakly improve upon a strong core allocation f when $\text{co}(\bigcup_{t \in T} \varphi_t^f \cap X_{L,\rho}) \cap H^f \neq \emptyset$. Since $X_{L,\rho}$ is a finite set and $0 \in \mathbb{R}^L$ is a relatively interior point of $\text{co}(\{f(a) - e(a) \mid a \in A\})$, $0 \in \mathbb{R}^L$ can be written by a convex combination of points $\bigcup_{t \in T} \varphi_t^f \cap X_{L,\rho}$ and $\{f(a) - e(a) \mid a \in A\}$, whose coefficients are all rational numbers with *bounded* denominators. This is a direct consequence of Lemma 5 in the appendix. We can make the upper bound of denominators dependent only on exogenous variables. From this convex combination, we can construct a coalition that can weakly improve upon f . Thus, for every strong core allocation f , $\text{co}(\bigcup_{t \in T} \varphi_t^f \cap X_{L,\rho}) \cap H^f = \emptyset$. The common denominator of the coefficients of the convex combination means the size of the coalition. In order to construct another allocation within the coalition that can weakly improve upon f , some agent $a \in A_t$ may have to consume the commodity bundle $f(b)$ of another agent b of type t . Here, the weak equal treatment property $f(a) \sim_t f(b)$ is working. We now turn to the formal proof.

Proof. Let $r \geq 1$ and $k \in \mathbb{Z}$ with $k \geq 2$. Let $T \subset \mathcal{P}_k \times \mathbb{Z}_+^L$ be a type set where $\#T \leq r$,

$\sum_{t \in T} e_t \gg 0$, and for some $t \in T$, \succsim_t is strongly monotone. Let

$$M = \max\{\|e\|_\infty \mid (\succsim, e) \in T\} \quad \text{and}$$

$$\xi = \max\{rM^2L^2(ML + 1), (kL + 1)ML\}.$$

Note that the number ξ depends only on exogenous variables. We first show that strong core allocations are uniformly bounded. Bewley [3, Theorem 1] proved that in an economy with the consumption set \mathbb{R}_+^L , strong core allocations are uniformly bounded. His proof uses a contradiction argument, so the bounds of strong core allocations are not clear. On the other hand, the method of proof of Mas-Colell [11, Lemma 7.4.10] can clarify the bounds of strong core allocations. We prove the next lemma with an argument similar to Mas-Colell's.

Lemma 1 ¹³ *For every finite set A of agents, if $\mathcal{E} : A \rightarrow T$ and $\#A_t/\#A \geq 1/r$ for all $t \in T$, then $\|f(a)\|_\infty \leq \xi$ for all $f \in C_S(\mathcal{E})$ and all $a \in A$.*

Proof. Let A be the set of agents and $\mathcal{E} : A \rightarrow T$ be an economy such that $\#A_t/\#A \geq 1/r$ for all $t \in T$. Let $f \in C_S(\mathcal{E})$. By a simple calculation, it follows that if $\#A \leq rML^2(ML + 1)$, then $\|f(a)\|_\infty \leq \xi$ for all $a \in A$. Therefore, in the remainder of the proof, we may assume that $\#A > rML^2(ML + 1)$. Let $J = \{j \in \{1, \dots, L\} \mid f^{(j)}(a) \geq ML + 1 \text{ for some } a \in A\}$ and $J' = \{j \in \{1, \dots, L\} \mid f^{(j)}(a) \geq (kL + 1)ML \text{ for some } a \in A\}$. Then, $J' \subset J$. If $J = \emptyset$, the proof has been completed. Thus, the set J is supposed to be nonempty. For $j \in J$, we choose $a_j \in \operatorname{argmax}\{f^{(j)}(a) \mid a \in A\}$. Note that $a_i = a_j$ may hold for some distinct indices i and j . Let $B = \{a_j \mid j \in J\}$. Then, $\#B \leq \#J \leq L$. The excess demand of the coalition B is denoted by $y = \sum_{a \in B} (f(a) - e(a)) \in \mathbb{Z}^L$. Let $J'' = \{j \in \{1, \dots, L\} \mid y^{(j)} \leq -1\}$. By a simple calculation, $y^{(j)} \geq 1$ for all $j \in J$. Thus, $J \cap J'' = \emptyset$. We also have:

$$(1) \quad y^{(j)} \geq kML^2 \text{ for all } j \in J'.$$

In addition, if $J' = \emptyset$, then $\|f(a)\|_\infty \leq \xi$ for all $a \in A$. Therefore, the proof is completed if we can show $J' = \emptyset$.

¹³From the following proof, we see that this lemma holds also for core allocations analyzed by Inoue [8].

Claim 1 $J' = \emptyset$.

Proof. Suppose that $J' \neq \emptyset$. Let $C = A \setminus B$. Then, $\#C = \#A - \#B \geq \#A - L$. Since $\#A > rML^2(ML + 1) > L$, we have $C \neq \emptyset$. We define $\tilde{y} \in \mathbb{Z}^L$ by

$$\tilde{y} = y - \sum_{j \in J''} y^{(j)} \chi_j.$$

Clearly, $\tilde{y} \geq 0$. Moreover, from (1), it follows that for all $j \in J'$, $\tilde{y}^{(j)} = y^{(j)} \geq kML^2$.

Subclaim 1.1 $J'' \neq \emptyset$.

Proof. Suppose that $J'' = \emptyset$. We have

$$\begin{aligned} kML^2 \sum_{j \in J'} \chi_j &\leq \tilde{y} = y \\ &= \sum_{a \in B} (f(a) - e(a)) \\ &= - \sum_{a \in C} (f(a) - e(a)) \\ &= \sum_{a \in C} (e(a) - f(a)). \end{aligned}$$

The third equality follows from the exact feasibility of strong core allocation f . Since $C \neq \emptyset$, we can choose an agent a^* of C . We define a mapping $g : C \rightarrow \mathbb{Z}_+^L$ by

$$g(a) = \begin{cases} f(a^*) + \sum_{c \in C} (e(c) - f(c)) & \text{if } a = a^*, \\ f(a) & \text{if } a \in C \setminus \{a^*\}. \end{cases}$$

Because $g(a^*) \geq f(a^*) + k\chi_j$ for all $j \in J'$ and $\succsim_{a^*} \in \mathcal{P}_k$, we have $g(a^*) \succ_{a^*} f(a^*)$. We also have

$$\sum_{a \in C} g(a) = \sum_{a \in C} f(a) + \sum_{c \in C} (e(c) - f(c)) = \sum_{a \in C} e(a).$$

This contradicts $f \in C_S(\mathcal{E})$. Thus, we have established the proof of Subclaim 1.1. \blacksquare

For all $j \in J''$, let $C_j = \{a \in C \mid f^{(j)}(a) \geq 1\}$.

Subclaim 1.2 $\#C_j > ML^2$ for all $j \in J''$.

Proof. Let $j \in J''$. Since $J \cap J'' = \emptyset$, it follows that $j \notin J$. Thus, $f^{(j)}(a) \leq ML$ for all $a \in A$. Since $C \setminus C_j = \{a \in C \mid f^{(j)}(a) < 1\} = \{a \in C \mid f^{(j)}(a) = 0\}$, we have

$$\begin{aligned} \sum_{a \in A} f^{(j)}(a) &\leq \{\#A - (\#C - \#C_j)\}ML \\ &= (\#C_j)ML + (\#A - \#C)ML \\ &\leq (\#C_j)ML + ML^2. \end{aligned}$$

On the other hand, since $\sum_{t \in T} e_t \gg 0$, there exists a type $t_j \in T$ such that $e_{t_j}^{(j)} \geq 1$.

Thus,

$$\frac{\#A}{r} \leq \#A_{t_j} \leq \sum_{a \in A_{t_j}} e^{(j)}(a) \leq \sum_{a \in A} e^{(j)}(a).$$

Because strong core allocation f is exactly feasible, we have

$$\frac{\#A}{r} \leq \sum_{a \in A} e^{(j)}(a) = \sum_{a \in A} f^{(j)}(a) \leq (\#C_j)ML + ML^2.$$

Thus,

$$\#C_j \geq \frac{\#A}{rML} - L > \frac{rML^2(ML + 1)}{rML} - L = ML^2.$$

This completes the proof of Subclaim 1.2. \blacksquare

For all $j \in J''$, we have

$$\begin{aligned} -1 \geq y^{(j)} &= \sum_{a \in B} (f^{(j)}(a) - e^{(j)}(a)) \\ &\geq -\sum_{a \in B} e^{(j)}(a) \\ &\geq -M(\#B) \\ &\geq -ML. \end{aligned}$$

Therefore, by Subclaim 1.2, there exists $\{G_j \mid j \in J''\}$ such that

$$\begin{aligned} G_j &\subset C_j && \text{for all } j \in J'', \\ \#G_j &= -y^{(j)} && \text{for all } j \in J'', \text{ and} \\ G_j \cap G_\ell &= \emptyset && \text{if } j \neq \ell. \end{aligned}$$

Since $J' \neq \emptyset$, by (1), there exists an $h \in \{1, \dots, L\}$ such that $\tilde{y}^{(h)} = y^{(h)} \geq kML^2$. We define a mapping $\hat{g} : C \rightarrow \mathbb{Z}^L$ by

$$\hat{g}(a) = \begin{cases} f(a) + k\chi_h - \chi_j & \text{if } a \in G_j \ (j \in J''), \\ f(a) & \text{if } a \in C \setminus \bigcup_{j \in J''} G_j. \end{cases}$$

For all $a \in G_j$, $f^{(j)}(a) \geq 1$ holds since $G_j \subset C_j$. Thus, $\hat{g}(a) \in \mathbb{Z}_+^L$ for all $a \in C$. Because $\succsim_a \in \mathcal{P}_k$ for all $a \in C$, we have $\hat{g}(a) \succ_a f(a)$ for all $a \in \bigcup_{j \in J''} G_j$. We also have

$$\begin{aligned}
\sum_{a \in C} \hat{g}(a) &= \sum_{a \in C} f(a) + k \sum_{j \in J''} (\#G_j) \chi_h - \sum_{j \in J''} (\#G_j) \chi_j \\
&= \sum_{a \in C} f(a) - k \left(\sum_{j \in J''} y^{(j)} \right) \chi_h + \sum_{j \in J''} y^{(j)} \chi_j \\
&\leq \sum_{a \in C} f(a) + kML^2 \chi_h + \sum_{j \in J''} y^{(j)} \chi_j \\
&\leq \sum_{a \in C} f(a) + \tilde{y} + \sum_{j \in J''} y^{(j)} \chi_j \\
&= \sum_{a \in C} f(a) + y \\
&= \sum_{a \in C} f(a) + \sum_{a \in B} (f(a) - e(a)) \\
&= \sum_{a \in A} f(a) - \sum_{a \in B} e(a) \\
&= \sum_{a \in A} e(a) - \sum_{a \in B} e(a) \\
&= \sum_{a \in C} e(a).
\end{aligned}$$

Although \hat{g} may not be exactly feasible within coalition C , coalition C can weakly improve upon f because agents' preference relations are weakly monotone. This contradicts $f \in C_S(\mathcal{E})$. This completes the proof of Claim 1. ■

Thus, we have established the proof of Lemma 1. ■

Next, we show that in an economy with a large number of agents, every strong core allocation has the weak equal treatment property. Before stating Lemma 2, we introduce some notation. Let $X_{L,\xi} = \{x \in \mathbb{Z}^L \mid \|x\|_\infty \leq \xi\}$. A set $\mathcal{X}_{L,\xi}$ is defined as follows. A mapping $\alpha : X_{L,\xi} \rightarrow \mathbb{Z}_+$ belongs to $\mathcal{X}_{L,\xi}$ if and only if $\sum_{x \in X_{L,\xi}} \alpha(x) \geq 1$, $\sum_{x \in X_{L,\xi}} \alpha(x)x = 0$, and there exists no mapping $\beta : X_{L,\xi} \rightarrow \mathbb{Z}_+$ such that

$$\begin{aligned}
\sum_{x \in X_{L,\xi}} \beta(x) &\geq 1, \quad \sum_{x \in X_{L,\xi}} \beta(x)x = 0, \\
\beta(x) &\leq \alpha(x) \quad \text{for all } x \in X_{L,\xi}, \text{ and} \\
\beta(y) &< \alpha(y) \quad \text{for some } y \in X_{L,\xi}.
\end{aligned}$$

Let

$$\mu(L, \xi) = \sup \left\{ \sum_{x \in X_{L, \xi}} \alpha(x) \mid \alpha \in \mathcal{X}_{L, \xi} \right\}.$$

The important thing is that $\mu(L, \xi)$ is finite, and therefore, it is a natural number. This fact is shown in Lemma 3 in the appendix.

Lemma 2¹⁴ *Let A be the set of agents such that $\#A > r\mu(L, \xi)$. If $\mathcal{E} : A \rightarrow T$ and $\#A_t/\#A \geq 1/r$ for all $t \in T$, then all strong core allocations for \mathcal{E} have the weak equal treatment property, i.e., for all $f \in C_S(\mathcal{E})$, all $t \in T$, and all $a, b \in A_t$, we have $f(a) \sim_t f(b)$.*

Proof. Suppose that there exists an allocation $f \in C_S(\mathcal{E})$, a type $t \in T$, and two agents $a, b \in A_t$ such that $f(a) \succ_t f(b)$. Without loss of generality, we can assume that $f(a) \succsim_t f(c) \succsim_t f(b)$ for all $c \in A_t$. By Lemma 1, $\|f(c)\|_\infty \leq \xi$ for all $c \in A$. Since both $f(c)$ and $e(c)$ are nonnegative vectors, we have $\|f(c) - e(c)\|_\infty \leq \max\{\xi, M\} = \xi$ for all $c \in A$. Thus, $f(c) - e(c) \in X_{L, \xi}$ for all $c \in A$. Define $\alpha : X_{L, \xi} \rightarrow \mathbb{Z}_+$ by, for all $x \in X_{L, \xi}$,

$$\alpha(x) = \#\{c \in A \mid f(c) - e(c) = x\}.$$

Note that

$$\begin{aligned} \sum_{x \in X_{L, \xi}} \alpha(x)x &= \sum_{c \in A} (f(c) - e(c)) = 0 \quad \text{and} \\ \sum_{x \in X_{L, \xi}} \alpha(x) &= \#A > r\mu(L, \xi). \end{aligned}$$

Thus, by the definition of $\mu(L, \xi)$, there exists a natural number $\ell > r$ and $\beta_j \in \mathcal{X}_{L, \xi}$ ($j = 1, \dots, \ell$) such that for all $x \in X_{L, \xi}$, $\alpha(x) = \sum_{j=1}^{\ell} \beta_j(x)$. Therefore, there exists a partition $\{B_1, \dots, B_\ell\}$ of A such that for all $j \in \{1, \dots, \ell\}$ and all $x \in X_{L, \xi}$,

$$\beta_j(x) = \#\{c \in B_j \mid f(c) - e(c) = x\}.$$

Without loss of generality, we can assume $b \in B_1$. Since $\beta_1 \in \mathcal{X}_{L, \xi}$, we have

$$\begin{aligned} \sum_{c \in B_1} (f(c) - e(c)) &= \sum_{x \in X_{L, \xi}} \beta_1(x)x = 0 \quad \text{and} \\ \#B_1 &= \sum_{x \in X_{L, \xi}} \beta_1(x) \leq \mu(L, \xi). \end{aligned}$$

¹⁴From the following proof, we see that this lemma holds also for core allocations analyzed by Inoue [8].

In addition, the set $A_t \setminus B_1$ is nonempty because $\#A_t \geq \#A/r > \mu(L, \xi) \geq \#B_1$.

Claim 2 $f(c) \sim_t f(b)$ for all $c \in A_t \setminus B_1$.

Proof. Suppose that $f(c^*) \succ_t f(b)$ for some $c^* \in A_t \setminus B_1$. We consider a coalition $C_1 = (A \setminus (B_1 \cup \{c^*\})) \cup \{b\}$. Define $g_1 : C_1 \rightarrow \mathbb{Z}_+^L$ by

$$g_1(c) = \begin{cases} f(c^*) & \text{if } c = b, \\ f(c) & \text{if } c \in C_1 \setminus \{b\}. \end{cases}$$

Since $\sum_{c \in A \setminus B_1} (f(c) - e(c)) = 0$ and agents b and c^* have the same type, it follows that $\sum_{c \in C_1} (g_1(c) - e(c)) = 0$. In addition, it follows that $g_1(b) = f(c^*) \succ_t f(b)$. This contradicts $f \in C_S(\mathcal{E})$. This completes the proof of Claim 2. ■

From Claim 2, it follows that $a \in B_1$. Since $A_t \setminus B_1$ is nonempty, we can pick $c' \in A_t \setminus B_1$. We consider a coalition $C_2 = (B_1 \setminus \{a\}) \cup \{c'\}$. Define $g_2 : C_2 \rightarrow \mathbb{Z}_+^L$ by

$$g_2(c) = \begin{cases} f(a) & \text{if } c = c', \\ f(c) & \text{if } c \in C_2 \setminus \{c'\}. \end{cases}$$

Since $\sum_{c \in B_1} (f(c) - e(c)) = 0$ and agents a and c' have the same type, it follows that $\sum_{c \in C_2} (g_2(c) - e(c)) = 0$. In addition, from Claim 2, it follows that $g_2(c') = f(a) \succ_t f(b) \sim_t f(c')$. This contradicts $f \in C_S(\mathcal{E})$. This completes the proof of Lemma 2. ■

Notice that completeness and transitivity of preference relations are essential in the proof of Lemma 2. This differs greatly from the case of an economy with infinitely many agents. In fact, in an atomless economy with the consumption set \mathbb{R}_+^L , Aumann [2] obtained a core equivalence theorem without completeness, transitivity, and convexity of preference relations. Therefore, in an atomless economy, a core equivalence theorem holds even though agents' preference relations are too weak to guarantee nonempty demand sets.

Let

$$\begin{aligned} \rho &= \xi + k(L-1)\xi + k \quad \text{and} \\ q &= L^{\frac{L}{2}} (2\rho)^L \{1 + L(\mu(L, \xi) - 1)\}. \end{aligned}$$

Note that both numbers ρ and q depend only on exogenous variables. Let A be the set of agents such that $\#A \geq rq$ and let $\mathcal{E} : A \rightarrow T$ be an economy such that $\#A_t/\#A \geq 1/r$ for all $t \in T$. We prove $C_S(\mathcal{E}) \subset W^*(\mathcal{E})$. Let $f \in C_S(\mathcal{E})$.

Claim 3 $\rho \chi_i \succ_a f(a)$ for all $i \in \{1, \dots, L\}$ and all $a \in A$.

Proof. Let $i \in \{1, \dots, L\}$ and $a \in A$. Since $\succsim_a \in \mathcal{P}_k$, it follows that

$$f(a) \prec_a f(a) - \sum_{j \neq i} f^{(j)}(a) \chi_j + k \left(\sum_{j \neq i} f^{(j)}(a) + 1 \right) \chi_i = \left(f^{(i)}(a) + k \sum_{j \neq i} f^{(j)}(a) + k \right) \chi_i.$$

By Lemma 1, we have $f^{(i)}(a) + k \sum_{j \neq i} f^{(j)}(a) + k \leq \xi + k(L-1)\xi + k = \rho$. Because of the weak monotonicity of preference relations, we have $f(a) \prec_a \rho \chi_i$. This completes the proof of Claim 3. ■

Let $H = \text{span}\{f(a) - e(a) \mid a \in A\}$. For all $t \in T$, define

$$\varphi_t = \{z \in \mathbb{Z}^L \mid z + e_t \in \mathbb{Z}_+^L \text{ and } z + e_t \succ_t f(a)\}$$

where $a \in A_t$. By Lemma 2, φ_t is well defined for all $t \in T$. For all $t \in T$, we define a subset φ'_t of φ_t as follows: $z \in \varphi'_t$ if and only if $z \in \varphi_t$ and there exists no $y \in \varphi_t$ with $y < z$. Thus, $\varphi_t \subset \varphi'_t + \mathbb{Z}_+^L$. Since agents' preference relations are weakly monotone, $\varphi_t + \mathbb{Z}_+^L = \varphi_t$ for all $t \in T$. Thus, $\varphi_t = \varphi'_t + \mathbb{Z}_+^L$ for all $t \in T$.

Claim 4 $\varphi'_t \subset X_{L,\rho} := \{x \in \mathbb{Z}^L \mid \|x\|_\infty \leq \rho\}$ for all $t \in T$.

Proof. Suppose that there exists a type $t \in T$ such that $\varphi'_t \not\subset X_{L,\rho}$. Since $\varphi'_t \subset \mathbb{Z}_+^L - \{e_t\} \subset \mathbb{Z}_+^L - \{(M, \dots, M)\}$ and $M < \rho$, we have $z^{(h)} > -\rho$ for all $z \in \varphi'_t$ and all $h \in \{1, \dots, L\}$. Therefore, from $\varphi'_t \not\subset X_{L,\rho}$, there exists a $z \in \varphi'_t$ and an $h \in \{1, \dots, L\}$ such that $z^{(h)} > \rho$. Since $\rho \chi_h \succ_t f(a)$ for all $a \in A_t$ by Claim 3, it follows that $\rho \chi_h - e_t \in \varphi_t$. We have

$$z^{(h)} > \rho \geq \rho \chi_h^{(h)} - e_t^{(h)}.$$

Since $z \geq -e_t$, it follows that for all i with $i \neq h$,

$$z^{(i)} \geq -e_t^{(i)} = \rho \chi_h^{(i)} - e_t^{(i)}.$$

Thus, $\rho \chi_h - e_t \in \varphi_t$ and $z > \rho \chi_h - e_t$. This contradicts $z \in \varphi'_t$. Therefore, we have established the proof of Claim 4. ■

By Claim 4, for all $t \in T$, we have

$$\begin{aligned}
\varphi_t &= \varphi'_t + \mathbb{Z}_+^L \\
&= (\varphi'_t \cap X_{L,\rho}) + \mathbb{Z}_+^L \\
&\subset (\varphi_t \cap X_{L,\rho}) + \mathbb{Z}_+^L \\
&\subset \varphi_t + \mathbb{Z}_+^L \\
&= \varphi_t.
\end{aligned}$$

Thus, $\varphi_t = (\varphi_t \cap X_{L,\rho}) + \mathbb{Z}_+^L$ for all $t \in T$. Therefore, $\bigcup_{t \in T} \varphi_t = (\bigcup_{t \in T} \varphi_t \cap X_{L,\rho}) + \mathbb{Z}_+^L$.

Claim 5 $\text{co} \left(\bigcup_{t \in T} \varphi_t \right) \cap H = \emptyset$ if and only if $\text{co} \left(\bigcup_{t \in T} \varphi_t \cap X_{L,\rho} \right) \cap H = \emptyset$.

Proof. It suffices to show the ‘if’ part only. We assume that $\text{co} \left(\bigcup_{t \in T} \varphi_t \cap X_{L,\rho} \right) \cap H = \emptyset$. Since the set $\bigcup_{t \in T} \varphi_t \cap X_{L,\rho}$ is finite, its convex hull $\text{co} \left(\bigcup_{t \in T} \varphi_t \cap X_{L,\rho} \right)$ is compact. Because H is closed and convex, by the separation theorem, there exists a vector $p \in \mathbb{R}^L \setminus \{0\}$ and real numbers α and β such that for all $z \in \text{co} \left(\bigcup_{t \in T} \varphi_t \cap X_{L,\rho} \right)$ and all $y \in H$,

$$p \cdot z \geq \alpha > \beta \geq p \cdot y.$$

Since H is a linear subspace, it follows that $p \in H^\perp$, where H^\perp is the orthogonal complement of H . Therefore, $p \cdot z > 0$ for all $z \in \text{co} \left(\bigcup_{t \in T} \varphi_t \cap X_{L,\rho} \right)$.

We show $p \geq 0$. On the contrary, suppose that $p^{(h)} < 0$ for some $h \in \{1, \dots, L\}$. Recall that $\rho \chi_h \succ_t f(a)$ for all $t \in T$ and all $a \in A_t$. Since $e_t \geq 0$ and \succsim_t is weakly monotone, we have $\rho \chi_h + e_t \succ_t f(a)$. Thus, $\rho \chi_h \in \varphi_t \cap X_{L,\rho}$. Hence,

$$0 < p \cdot (\rho \chi_h) = p^{(h)} \rho.$$

This contradicts $p^{(h)} < 0$ and $\rho > 0$. Thus, we conclude $p \geq 0$. Since $\bigcup_{t \in T} \varphi_t = (\bigcup_{t \in T} \varphi_t \cap X_{L,\rho}) + \mathbb{Z}_+^L$, it follows that $p \cdot z > 0$ for all $z \in \text{co} \left(\bigcup_{t \in T} \varphi_t \right)$. Then, $\text{co} \left(\bigcup_{t \in T} \varphi_t \right) \cap H = \emptyset$. This completes the proof of Claim 5. ■

Claim 6 $\text{co} \left(\bigcup_{t \in T} \varphi_t \cap X_{L,\rho} \right) \cap H = \emptyset$.

Proof. Suppose that $\text{co} \left(\bigcup_{t \in T} \varphi_t \cap X_{L,\rho} \right) \cap H \neq \emptyset$. For all $t \in T$, we denote $\{f(a) - e(a) \mid a \in A_t\} = \{z_{t,j} \mid j = 1, \dots, \ell_t\}$. Note that, by Lemma 1, $z_{t,j} \in X_{L,\xi}$ for all t and all

j . Let $\eta_t^{(j)} = \#\{a \in A_t \mid f(a) - e(a) = z_{t,j}\} \in \mathbb{Z}_{++}$ for all $t \in T$ and all $j \in \{1, \dots, \ell_t\}$.

Since

$$0 = \sum_{a \in A} (f(a) - e(a)) = \sum_{t \in T} \sum_{a \in A_t} (f(a) - e(a)) = \sum_{t \in T} \sum_{j=1}^{\ell_t} \eta_t^{(j)} z_{t,j},$$

by Lemma 5 in the appendix, there are $q_0 \in \mathbb{Z}_{++}$ with $q_0 \leq q$, $\{x_{t,j} \mid j = 1, \dots, m_t\} \subset \varphi_t \cap X_{L,\rho}$ ($t \in T$), $(\alpha_t^{(1)}, \dots, \alpha_t^{(m_t)}) \in \mathbb{Q}_+^{m_t}$ ($t \in T$), and $(\beta_t^{(1)}, \dots, \beta_t^{(\ell_t)}) \in \mathbb{Q}_+^{\ell_t}$ ($t \in T$) such that

$$\begin{aligned} \sum_{t \in T} \sum_{j=1}^{m_t} \alpha_t^{(j)} &> 0, \\ \sum_{t \in T} \sum_{j=1}^{m_t} \alpha_t^{(j)} + \sum_{t \in T} \sum_{j=1}^{\ell_t} \beta_t^{(j)} &= 1, \\ q_0 \alpha_t^{(j)} &\in \mathbb{Z}_+ \quad \text{for all } t \in T \text{ and all } j \in \{1, \dots, m_t\}, \\ q_0 \beta_t^{(j)} &\in \mathbb{Z}_+ \quad \text{for all } t \in T \text{ and all } j \in \{1, \dots, \ell_t\}, \text{ and} \\ \sum_{t \in T} \sum_{j=1}^{m_t} \alpha_t^{(j)} x_{t,j} + \sum_{t \in T} \sum_{j=1}^{\ell_t} \beta_t^{(j)} z_{t,j} &= 0. \end{aligned}$$

Note that $q_0 \left(\sum_{j=1}^{m_t} \alpha_t^{(j)} + \sum_{j=1}^{\ell_t} \beta_t^{(j)} \right) \leq q_0 \leq q \leq \#A/r \leq \#A_t$ for all $t \in T$. Therefore, for all $t \in T$, there are families $\{C_{t,j} \mid j = 1, \dots, m_t\}$ and $\{D_{t,j} \mid j = 1, \dots, \ell_t\}$ of subsets of A_t such that

$$\begin{aligned} \{C_{t,j} \mid j = 1, \dots, m_t\} \cup \{D_{t,j} \mid j = 1, \dots, \ell_t\} &\text{ is pairwise disjoint,} \\ \#C_{t,j} &= q_0 \alpha_t^{(j)} \quad \text{for all } j \in \{1, \dots, m_t\}, \text{ and} \\ \#D_{t,j} &= q_0 \beta_t^{(j)} \quad \text{for all } j \in \{1, \dots, \ell_t\}. \end{aligned}$$

We now consider the coalition $S = \bigcup_{t \in T} \left(\bigcup_{j=1}^{m_t} C_{t,j} \cup \bigcup_{j=1}^{\ell_t} D_{t,j} \right)$. Since $\sum_{t \in T} \sum_{j=1}^{m_t} \alpha_t^{(j)} > 0$, the set $\bigcup_{t \in T} \bigcup_{j=1}^{m_t} C_{t,j}$ is not empty. Define a mapping $g : S \rightarrow \mathbb{Z}_+^L$ by

$$g(a) = \begin{cases} x_{t,j} + e_t & \text{if } a \in C_{t,j} \text{ } (t \in T, j = 1, \dots, m_t), \\ z_{t,j} + e_t & \text{if } a \in D_{t,j} \text{ } (t \in T, j = 1, \dots, \ell_t). \end{cases}$$

By the way of construction of S , we have $\sum_{a \in S} g(a) = \sum_{a \in S} e(a)$. For all $a \in C_{t,j}$, since $x_{t,j} \in \varphi_t$, we have $g(a) = x_{t,j} + e_t \succ_t f(a)$. For all $a \in D_{t,j}$, there exists $b \in A_t$ such that $g(a) = z_{t,j} + e_t = f(b)$. Thus, for all $a \in D_{t,j}$, $g(a) = f(b) \sim_t f(a)$ follows from Lemma 2. This contradicts $f \in C_S(\mathcal{E})$. Therefore, we have established the proof of Claim 6. \blacksquare

From Claims 5 and 6, it follows that $\text{co} \left(\bigcup_{t \in T} \varphi_t \right) \cap H = \emptyset$. Since $\bigcup_{t \in T} \varphi_t \subset \mathbb{Z}_+^L - \{(M, \dots, M)\}$ and $\bigcup_{t \in T} \varphi_t + \mathbb{Z}_+^L = \bigcup_{t \in T} \varphi_t$, by Lemma 6 in the appendix, there exists a

vector $p \in H^\perp \cap \mathbb{Z}_+^L$ and a positive real number ε such that for all $z \in \text{co}(\bigcup_{t \in T} \varphi_t)$,

$$p \cdot z \geq \varepsilon.$$

Since $f(a) - e(a) \in H$ and $p \in H^\perp$, we have $p \cdot (f(a) - e(a)) = 0$. If $a \in A_t$, $x \in \mathbb{Z}_+^L$, and $x \succ_t f(a)$, then $x - e(a) \in \varphi_t$. Hence, we have $p \cdot (x - e(a)) \geq \varepsilon > 0$. Therefore, a pair (p, f) is a Walras equilibrium. Thus, $f \in W^*(\mathcal{E})$. This completes the proof of the theorem. ■

Appendix

The following results are used in the proof of the theorem.

Lemma 3 *Let L and N be natural numbers. Let $X_{L,N} = \{x \in \mathbb{Z}^L \mid \|x\|_\infty \leq N\}$. Define a set $\mathcal{X}_{L,N}$ as follows: A mapping $\alpha : X_{L,N} \rightarrow \mathbb{Z}_+$ belongs to $\mathcal{X}_{L,N}$ if and only if $\sum_{x \in X_{L,N}} \alpha(x) \geq 1$, $\sum_{x \in X_{L,N}} \alpha(x)x = 0$, and there exists no mapping $\beta : X_{L,N} \rightarrow \mathbb{Z}_+$ such that $\sum_{x \in X_{L,N}} \beta(x) \geq 1$, $\sum_{x \in X_{L,N}} \beta(x)x = 0$, $\beta(x) \leq \alpha(x)$ for all $x \in X_{L,N}$, and $\beta(y) < \alpha(y)$ for some $y \in X_{L,N}$. Then, the number*

$$\mu(L, N) := \sup \left\{ \sum_{x \in X_{L,N}} \alpha(x) \mid \alpha \in \mathcal{X}_{L,N} \right\}$$

is finite. Hence, $\mu(L, N)$ can be achieved by some $\alpha \in \mathcal{X}_{L,N}$. In particular,

$$\begin{aligned} \mu(1, 1) &= 2, \\ \mu(1, N) &\leq \frac{1}{2} N^2(N+1) \quad \text{for all } N \geq 2, \text{ and} \\ \mu(L+1, N) &\leq \mu(L, N)\mu(1, N(N+1)\mu(L, N)) \quad \text{for all } L, N \in \mathbb{Z}_{++}. \end{aligned}$$

Proof. We show the lemma by induction on L . Let $L = 1$. Clearly, $\mu(1, 1) = 2$. We consider the case where $N \geq 2$.

Claim 7 $\mu(1, N) \leq \frac{1}{2} N^2(N+1)$ for all $N \geq 2$.

Proof. Suppose that there exists a mapping $\alpha \in \mathcal{X}_{1,N}$ such that $\sum_{x \in X_{1,N}} \alpha(x) > N^2(N+1)/2$. Because $\sum_{x \in X_{1,N}} \alpha(x) > 2$, by the definition of $\mathcal{X}_{1,L}$, $\alpha(0) = 0$ and there exists no

$\ell \in X_{1,N} \setminus \{0\}$ such that $\alpha(\ell) \geq 1$ and $\alpha(-\ell) \geq 1$. Thus, $\#\{x \in X_{1,N} \mid \alpha(x) \geq 1\} \leq N$. Since $\sum_{x \in X_{1,N}} \alpha(x) > N^2(N+1)/2$, there exists an integer $m \in X_{1,N} \setminus \{0\}$ such that

$$\alpha(m) > \frac{1}{2}N(N+1).$$

We may assume that $m \geq 1$.

Subclaim 7.1 If $\ell \in \mathbb{Z}$ and $1 \leq \ell \leq N$, then $\alpha(-\ell) < m$.

Proof. Suppose that there exists an $\ell \in \mathbb{Z}$ such that $1 \leq \ell \leq N$ and $\alpha(-\ell) \geq m$. Note that $\alpha(m) > N(N+1)/2 > N \geq \ell$. We define a mapping $\beta : X_{1,N} \rightarrow \mathbb{Z}_+$ by

$$\beta(x) = \begin{cases} m & \text{if } x = -\ell, \\ \ell & \text{if } x = m, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\begin{aligned} \sum_{x \in X_{1,N}} \beta(x) &\geq 1, \\ \beta(x) &\leq \alpha(x) \quad \text{for all } x \in X_{1,N}, \text{ and} \\ \beta(m) &< \alpha(m). \end{aligned}$$

Moreover, we have $\sum_{x \in X_{1,N}} \beta(x)x = m(-\ell) + \ell m = 0$. This contradicts $\alpha \in \mathcal{X}_{1,N}$.

Therefore, we have established the proof of Subclaim 7.1. ■

Since

$$0 = \sum_{x \in X_{1,N}} \alpha(x)x = \sum_{\ell=1}^N \alpha(\ell)\ell + \sum_{\ell=1}^N \alpha(-\ell)(-\ell),$$

we have $\sum_{\ell=1}^N \alpha(-\ell)\ell = \sum_{\ell=1}^N \alpha(\ell)\ell$. On the other hand, from Subclaim 7.1, it follows that

$$\begin{aligned} \sum_{\ell=1}^N \alpha(-\ell)\ell &< m \sum_{\ell=1}^N \ell \\ &= m \times \frac{1}{2}N(N+1) \\ &< \alpha(m)m \\ &\leq \sum_{\ell=1}^N \alpha(\ell)\ell, \end{aligned}$$

which is a contradiction. This completes the proof of Claim 7. ■

Let $K \in \mathbb{Z}_{++}$. Assume that for all $N \in \mathbb{Z}_{++}$ and all $L \in \mathbb{Z}_{++}$ with $L \leq K$, $\mu(L, N)$ is finite. We now prove that $\mu(K+1, N)$ is finite for all $N \in \mathbb{Z}_{++}$.

Claim 8 $\mu(K+1, N) \leq \mu(K, N)\mu(1, N(N+1)\mu(K, N))$ for all $N \in \mathbb{Z}_{++}$.

Proof. Suppose that $\mu(K+1, N) > \mu(K, N)\mu(1, N(N+1)\mu(K, N))$ for some $N \in \mathbb{Z}_{++}$. Then, there exists a mapping $\alpha \in \mathcal{X}_{K+1, N}$ such that

$$\sum_{x \in X_{K+1, N}} \alpha(x) > \mu(K, N)\mu(1, N(N+1)\mu(K, N)).$$

We define a mapping $\beta : X_{K, N} \rightarrow \mathbb{Z}_+$ by $\beta(y) = \sum_{\ell \in X_{1, N}} \alpha(\ell, y)$ for all $y \in X_{K, N}$. From $\alpha \in \mathcal{X}_{K+1, N}$, it follows that

$$\sum_{y \in X_{K, N}} \beta(y)y = \sum_{y \in X_{K, N}} \sum_{\ell \in X_{1, N}} \alpha(\ell, y)y = 0.$$

We also have

$$\begin{aligned} \sum_{y \in X_{K, N}} \beta(y) &= \sum_{y \in X_{K, N}} \sum_{\ell \in X_{1, N}} \alpha(\ell, y) \\ &= \sum_{x \in X_{K+1, N}} \alpha(x) \\ &> \mu(K, N)\mu(1, N(N+1)\mu(K, N)). \end{aligned}$$

Thus, there exists a natural number $k > \mu(1, N(N+1)\mu(K, N))$ and mappings $\tilde{\gamma}_j \in \mathcal{X}_{K, N}$ ($j = 1, \dots, k$) such that for all $y \in X_{K, N}$, $\sum_{j=1}^k \tilde{\gamma}_j(y) = \beta(y)$. Since $\sum_{j=1}^k \tilde{\gamma}_j(y) = \sum_{\ell \in X_{1, N}} \alpha(\ell, y)$ for all $y \in X_{K, N}$, there are mappings $\gamma_j : X_{K+1, N} \rightarrow \mathbb{Z}_+$ ($j = 1, \dots, k$) such that

$$\begin{aligned} \sum_{\ell \in X_{1, N}} \gamma_j(\ell, y) &= \tilde{\gamma}_j(y) \quad \text{for all } y \in X_{K, N} \text{ and all } j \in \{1, \dots, k\} \text{ and} \\ \sum_{j=1}^k \gamma_j(\ell, y) &= \alpha(\ell, y) \quad \text{for all } (\ell, y) \in X_{K+1, N}. \end{aligned}$$

For every $j \in \{1, \dots, k\}$, let

$$\delta_j = \sum_{\ell \in X_{1, N}} \sum_{y \in X_{K, N}} \gamma_j(\ell, y)\ell \in \mathbb{Z}.$$

Since $\alpha \in \mathcal{X}_{K+1, N}$, we have

$$\sum_{j=1}^k \delta_j = \sum_{\ell \in X_{1, N}} \sum_{y \in X_{K, N}} \sum_{j=1}^k \gamma_j(\ell, y)\ell = \sum_{\ell \in X_{1, N}} \sum_{y \in X_{K, N}} \alpha(\ell, y)\ell = 0.$$

Since $\tilde{\gamma}_j \in \mathcal{X}_{K,N}$, it follows that for all $\ell \in X_{1,N}$,

$$\sum_{y \in X_{K,N}} \gamma_j(\ell, y) \leq \sum_{y \in X_{K,N}} \tilde{\gamma}_j(y) \leq \mu(K, N).$$

Therefore, for all $j \in \{1, \dots, k\}$,

$$|\delta_j| \leq \sum_{\ell \in X_{1,N}} |\ell| \mu(K, N) = N(N+1)\mu(K, N).$$

Thus, $\delta_j \in X_{1, N(N+1)\mu(K, N)}$ for all $j \in \{1, \dots, k\}$. Since $\sum_{j=1}^k \delta_j = 0$ and $k > \mu(1, N(N+1)\mu(K, N))$, there exists a subset J of $\{1, \dots, k\}$ such that $\emptyset \neq J \subsetneq \{1, \dots, k\}$ and $\sum_{j \in J} \delta_j = 0$. Define a mapping $\zeta : X_{K+1, N} \rightarrow \mathbb{Z}_+$ by, for all $x \in X_{K+1, N}$,

$$\zeta(x) = \sum_{j \in J} \gamma_j(x).$$

We have

$$\begin{aligned} \sum_{x \in X_{K+1, N}} \zeta(x)x^{(1)} &= \sum_{\ell \in X_{1, N}} \sum_{y \in X_{K, N}} \sum_{j \in J} \gamma_j(\ell, y)\ell \\ &= \sum_{j \in J} \delta_j \\ &= 0. \end{aligned}$$

Since $\tilde{\gamma}_j \in \mathcal{X}_{K, N}$ for all j , we have

$$\begin{aligned} \sum_{(\ell, y) \in X_{K+1, N}} \zeta(\ell, y)y &= \sum_{y \in X_{K, N}} \sum_{\ell \in X_{1, N}} \zeta(\ell, y)y \\ &= \sum_{j \in J} \sum_{y \in X_{K, N}} \sum_{\ell \in X_{1, N}} \gamma_j(\ell, y)y \\ &= \sum_{j \in J} \sum_{y \in X_{K, N}} \tilde{\gamma}_j(y)y \\ &= 0. \end{aligned}$$

Thus, $\sum_{x \in X_{K+1, N}} \zeta(x)x = 0$. Since $J \neq \emptyset$, we have $\sum_{x \in X_{K+1, N}} \zeta(x) \geq 1$. It is obvious that for all $x \in X_{K+1, N}$,

$$\zeta(x) = \sum_{j \in J} \gamma_j(x) \leq \sum_{j=1}^k \gamma_j(x) = \alpha(x).$$

Since $J \subsetneq \{1, \dots, k\}$ and $\sum_{x \in X_{K+1, N}} \gamma_j(x) \geq 1$ for all $j \in \{1, \dots, k\}$, there exists an element x^* of $X_{K+1, N}$ such that

$$\zeta(x^*) = \sum_{j \in J} \gamma_j(x^*) < \sum_{j=1}^k \gamma_j(x^*) = \alpha(x^*).$$

This contradicts $\alpha \in \mathcal{X}_{K+1,N}$. This completes the proof of Claim 8. ■

Hence, we establish the proof of Lemma 3. ■

Lemma 4 (Hadamard's inequality) *If $B = (b_1, \dots, b_\ell)$ is an $\ell \times \ell$ matrix of real numbers, then*

$$|\det B| \leq \prod_{j=1}^{\ell} \|b_j\|,$$

where $\det B$ is the determinant of matrix B and $\|\cdot\|$ is the Euclidean norm.

Proof. See Dunford and Schwartz [5, pp.1018-1019]. ■

Lemma 5 *Let L, K , and N be natural numbers. Let $\{z_1, \dots, z_s\} \subset X_{L,K}$ and $\sum_{j=1}^s \eta^{(j)} z_j = 0$ for some $(\eta^{(1)}, \dots, \eta^{(s)}) \in \mathbb{Z}_{++}^s$. Let $H = \text{span}\{z_1, \dots, z_s\}$ and*

$$q = L^{\frac{L}{2}} \max\{K^L, (2N)^L\} + L^{\frac{L+2}{2}} (\mu(L, K) - 1) \max\{K^{L-1}N, (2N)^L\}.$$

Then, for all $E \subset X_{L,N}$ with $\text{co}(E) \cap H \neq \emptyset$, there are $q_0 \in \mathbb{Z}_{++}$ with $q_0 \leq q$, $m \in \mathbb{Z}_+$, $\{x_0, \dots, x_m\} \subset E$, $(\alpha^{(0)}, \dots, \alpha^{(m)}) \in \mathbb{Q}_{++}^{m+1}$, and $(\beta^{(1)}, \dots, \beta^{(s)}) \in \mathbb{Q}_+^s$ such that

$$\begin{aligned} \sum_{j=0}^m \alpha^{(j)} + \sum_{j=1}^s \beta^{(j)} &= 1, \\ q_0 \alpha^{(j)} &\in \mathbb{Z}_{++} \quad \text{for all } j \in \{0, \dots, m\}, \\ q_0 \beta^{(j)} &\in \mathbb{Z}_+ \quad \text{for all } j \in \{1, \dots, s\}, \text{ and} \\ \sum_{j=0}^m \alpha^{(j)} x_j + \sum_{j=1}^s \beta^{(j)} z_j &= 0. \end{aligned}$$

Proof. Pick any $E \subset X_{L,N}$ with $\text{co}(E) \cap H \neq \emptyset$. We may assume that, for some $\ell < s$, $\{z_1, \dots, z_\ell\}$ is a basis of H , i.e., $\{z_1, \dots, z_\ell\}$ is linearly independent and $H = \text{span}\{z_1, \dots, z_\ell\}$. When $H = \{0\}$, we can obtain the required assertion from a slight modification of the following proof. We now consider two distinct cases.

Case 1 $E \cap H \neq \emptyset$.

We can choose an element y^* of $E \cap H$. Since $\{z_1, \dots, z_\ell\}$ is a basis of H , there exists a unique vector $(\theta^{(1)}, \dots, \theta^{(\ell)}) \in \mathbb{R}^\ell$ such that $y^* = \sum_{j=1}^{\ell} \theta^{(j)} z_j$. Without loss of generality, we may assume that the family $\{z_1, \dots, z_\ell, \chi_{\ell+1}, \dots, \chi_L\}$ of vectors is a basis of \mathbb{R}^L , where χ_i is the i th unit vector. Let $\hat{y}^* = (y^{*(1)}, \dots, y^{*(\ell)})^T \in \mathbb{Z}^\ell$ and let $\hat{z}_j = (z_j^{(1)}, \dots, z_j^{(\ell)})^T \in$

\mathbb{Z}^ℓ for all $j \in \{1, \dots, \ell\}$, where the symbol T is the transposition operator of vectors. Therefore, \hat{y}^* and \hat{z}_j , $j \in \{1, \dots, \ell\}$, are column vectors. Note that the family $\{\hat{z}_1, \dots, \hat{z}_\ell\}$ of vectors is linearly independent. Let $B_1 = (\hat{z}_1, \dots, \hat{z}_\ell)$. Then, B_1 is an $\ell \times \ell$ matrix.

Since

$$B_1 \begin{pmatrix} \theta^{(1)} \\ \vdots \\ \theta^{(\ell)} \end{pmatrix} = \hat{y}^*,$$

from Cramer's rule, it follows that for all $j \in \{1, \dots, \ell\}$,

$$\theta^{(j)} = \frac{1}{\det B_1} \cdot \det(\hat{z}_1, \dots, \hat{z}_{j-1}, \hat{y}^*, \hat{z}_{j+1}, \dots, \hat{z}_\ell) \in \mathbb{Q}.$$

Note that $|\det B_1| \theta^{(j)} \in \mathbb{Z}$ for all $j \in \{1, \dots, \ell\}$. Using Hadamard's inequality (Lemma 4), we have

$$1 \leq |\det B_1| \leq \ell^{\frac{\ell}{2}} K^\ell \quad \text{and}$$

$$|\det B_1| |\theta^{(j)}| = |\det(\hat{z}_1, \dots, \hat{z}_{j-1}, \hat{y}^*, \hat{z}_{j+1}, \dots, \hat{z}_\ell)| \leq \ell^{\frac{\ell}{2}} K^{\ell-1} N \quad \text{for all } j \in \{1, \dots, \ell\}.$$

Define a mapping $\bar{\eta} : X_{L,K} \rightarrow \mathbb{Z}_+$ by

$$\bar{\eta}(x) = \begin{cases} \eta^{(j)} & \text{if } x = z_j \text{ (} j = 1, \dots, s), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\bar{\eta}(z_j) = \eta^{(j)} \gg 0$ for all $j \in \{1, \dots, s\}$ and $\sum_{x \in X_{L,K}} \bar{\eta}(x)x = \sum_{j=1}^s \eta^{(j)} z_j = 0$, by the definition of $\mathcal{X}_{L,K}$, for all $j \in \{1, \dots, s\}$, there exists a $\bar{\lambda}_j \in \mathcal{X}_{L,K}$ such that $\bar{\lambda}_j(z_j) \geq 1$ and for all $x \in X_{L,K}$, $\bar{\lambda}_j(x) \leq \bar{\eta}(x)$. Note that for all $j \in \{1, \dots, s\}$,

$$\begin{aligned} \sum_{i=1}^s \bar{\lambda}_j(z_i) z_i &= \sum_{x \in X_{L,K}} \bar{\lambda}_j(x) x = 0 \quad \text{and} \\ \sum_{i=1}^s \bar{\lambda}_j(z_i) &= \sum_{x \in X_{L,K}} \bar{\lambda}_j(x) \leq \mu(L, K). \end{aligned}$$

For every $i, j \in \{1, \dots, s\}$, define a $\lambda_j^{(i)}$ by

$$\lambda_j^{(i)} = \begin{cases} \bar{\lambda}_j(z_i) & \text{if } i \neq j, \\ \bar{\lambda}_j(z_j) - 1 & \text{if } i = j. \end{cases}$$

Note that for all $i, j \in \{1, \dots, s\}$, $\lambda_j^{(i)} \in \mathbb{Z}_+$. Therefore, for all $j \in \{1, \dots, s\}$, we have

$$\begin{aligned} \sum_{i=1}^s \lambda_j^{(i)} z_i &= \sum_{i=1}^s \bar{\lambda}_j(z_i) z_i - z_j = -z_j \quad \text{and} \\ \sum_{i=1}^s \lambda_j^{(i)} &= \sum_{i=1}^s \bar{\lambda}_j(z_i) - 1 \leq \mu(L, K) - 1. \end{aligned}$$

Let $J_1 = \{j \in \{1, \dots, \ell\} \mid \theta^{(j)} > 0\}$, $J_2 = \{1, \dots, \ell\} \setminus J_1$, and $J_3 = \{\ell + 1, \dots, s\}$. We have

$$\begin{aligned}
0 &= y^* + \sum_{j \in J_1} -\theta^{(j)} z_j + \sum_{j \in J_2} -\theta^{(j)} z_j \\
&= y^* + \sum_{j \in J_1} \theta^{(j)} \sum_{i=1}^s \lambda_j^{(i)} z_i + \sum_{j \in J_2} -\theta^{(j)} z_j \\
&= y^* + \sum_{i \in J_1 \cup J_3} \sum_{j \in J_1} \theta^{(j)} \lambda_j^{(i)} z_i + \sum_{i \in J_2} \left(-\theta^{(i)} + \sum_{j \in J_1} \theta^{(j)} \lambda_j^{(i)} \right) z_i.
\end{aligned}$$

For every $i \in \{1, \dots, s\}$, define

$$\pi^{(i)} = \begin{cases} \sum_{j \in J_1} \theta^{(j)} \lambda_j^{(i)} & \text{if } i \in J_1 \cup J_3, \\ -\theta^{(i)} + \sum_{j \in J_1} \theta^{(j)} \lambda_j^{(i)} & \text{if } i \in J_2. \end{cases}$$

Note that $|\det B_1| \pi^{(i)} \in \mathbb{Z}_+$ for all $i \in \{1, \dots, s\}$. Let

$$\begin{aligned}
\alpha &= \frac{1}{1 + \sum_{j=1}^s \pi^{(j)}} \in \mathbb{Q}_{++} \quad \text{and} \\
\beta^{(i)} &= \frac{\pi^{(i)}}{1 + \sum_{j=1}^s \pi^{(j)}} \in \mathbb{Q}_+ \quad \text{for all } i \in \{1, \dots, s\}.
\end{aligned}$$

Then, $\alpha + \sum_{i=1}^s \beta^{(i)} = 1$ and $0 = \alpha y^* + \sum_{i=1}^s \beta^{(i)} z_i$. Let $q_1 = |\det B_1| (1 + \sum_{j=1}^s \pi^{(j)})$.

Then, it follows that $q_1 \in \mathbb{Z}_{++}$, $q_1 \alpha \in \mathbb{Z}_{++}$, and $q_1 \beta^{(i)} \in \mathbb{Z}_+$ for all $i \in \{1, \dots, s\}$. We

also have

$$\begin{aligned}
q_1 &= |\det B_1| \left\{ 1 + \sum_{j \in J_1} \theta^{(j)} \sum_{i=1}^s \lambda_j^{(i)} - \sum_{j \in J_2} \theta^{(j)} \right\} \\
&= |\det B_1| + \sum_{j \in J_1} |\det B_1| \theta^{(j)} \sum_{i=1}^s \lambda_j^{(i)} - \sum_{j \in J_2} |\det B_1| \theta^{(j)} \\
&\leq |\det B_1| + (\#J_1) \ell^{\frac{\ell}{2}} K^{\ell-1} N(\mu(L, K) - 1) + (\#J_2) \ell^{\frac{\ell}{2}} K^{\ell-1} N \\
&\leq \ell^{\frac{\ell}{2}} K^{\ell} + \ell^{\frac{\ell+2}{2}} K^{\ell-1} N(\mu(L, K) - 1) \\
&\leq q.
\end{aligned}$$

Therefore, we obtain the required assertion when Case 1 occurs.

Case 2 $E \cap H = \emptyset$.

Since the set $\text{co}(E) \cap H$ is nonempty and compact, there exists an extreme point y^* of $\text{co}(E) \cap H$. Since $\{z_1, \dots, z_\ell\}$ is a basis of H , there exists a unique vector $(\theta^{(1)}, \dots, \theta^{(\ell)}) \in$

\mathbb{R}^ℓ such that $y^* = \sum_{j=1}^\ell \theta^{(j)} z_j$. Since $y^* \in \text{co}(E) \setminus E$, there are affinely independent vectors $\{x_0, x_1, \dots, x_m\} \subset E$ and a vector $(\kappa^{(0)}, \kappa^{(1)}, \dots, \kappa^{(m)}) \in \mathbb{R}_{++}^{m+1}$ ($m \geq 1$) such that

$$\sum_{j=0}^m \kappa^{(j)} = 1 \quad \text{and} \quad y^* = \sum_{j=0}^m \kappa^{(j)} x_j = \sum_{j=1}^m \kappa^{(j)} (x_j - x_0) + x_0.$$

Claim 9 The family $\{z_1, \dots, z_\ell, x_1 - x_0, \dots, x_m - x_0\}$ of vectors is linearly independent.

Proof. Suppose that $\{z_1, \dots, z_\ell, x_1 - x_0, \dots, x_m - x_0\}$ is linearly dependent. Since both families $\{z_1, \dots, z_\ell\}$ and $\{x_1 - x_0, \dots, x_m - x_0\}$ are linearly independent, there are nonzero vectors $(\sigma^{(1)}, \dots, \sigma^{(\ell)}) \in \mathbb{R}^\ell \setminus \{0\}$ and $(\tau^{(1)}, \dots, \tau^{(m)}) \in \mathbb{R}^m \setminus \{0\}$ such that

$$\sum_{j=1}^\ell \sigma^{(j)} z_j + \sum_{j=1}^m \tau^{(j)} (x_j - x_0) = 0.$$

Since $0 = -y^* + \sum_{j=1}^m \kappa^{(j)} (x_j - x_0) + x_0 = -\sum_{j=1}^\ell \theta^{(j)} z_j + \sum_{j=1}^m \kappa^{(j)} (x_j - x_0) + x_0$, it follows that for all $t \in \mathbb{R}$,

$$\sum_{j=1}^\ell (t\sigma^{(j)} - \theta^{(j)}) z_j + \sum_{j=1}^m (t\tau^{(j)} + \kappa^{(j)}) (x_j - x_0) + x_0 = 0.$$

Since $\kappa^{(j)} > 0$ for all $j \in \{1, \dots, m\}$ and $\sum_{j=1}^m \kappa^{(j)} < 1$, there exists a real number $t_1 > 0$ such that

$$\begin{aligned} \kappa^{(j)} + t_1 \tau^{(j)} > 0 \quad \text{and} \quad \kappa^{(j)} - t_1 \tau^{(j)} > 0 \quad \text{for all } j \in \{1, \dots, m\}, \text{ and} \\ \sum_{j=1}^m (\kappa^{(j)} + t_1 \tau^{(j)}) < 1 \quad \text{and} \quad \sum_{j=1}^m (\kappa^{(j)} - t_1 \tau^{(j)}) < 1. \end{aligned}$$

Let

$$y_1 = \sum_{j=1}^m (\kappa^{(j)} + t_1 \tau^{(j)}) (x_j - x_0) + x_0 = \sum_{j=1}^\ell (\theta^{(j)} - t_1 \sigma^{(j)}) z_j$$

and

$$y_2 = \sum_{j=1}^m (\kappa^{(j)} - t_1 \tau^{(j)}) (x_j - x_0) + x_0 = \sum_{j=1}^\ell (\theta^{(j)} + t_1 \sigma^{(j)}) z_j.$$

Then, $y_1, y_2 \in \text{co}(E) \cap H$. In addition, we have

$$\begin{aligned} \frac{1}{2} y_1 + \frac{1}{2} y_2 &= \sum_{j=1}^m \kappa^{(j)} (x_j - x_0) + x_0 = y^* \quad \text{and} \\ y_1 - y_2 &= 2t_1 \sum_{j=1}^m \tau^{(j)} (x_j - x_0) \neq 0, \end{aligned}$$

but this contradicts that y^* is an extreme point of $\text{co}(E) \cap H$. This completes the proof of Claim 9. ■

Without loss of generality, we may assume that $\{z_1, \dots, z_\ell, x_1 - x_0, \dots, x_m - x_0, \chi_{\ell+m+1}, \dots, \chi_L\}$ is a basis of \mathbb{R}^L . Let $\bar{z}_j = (z_j^{(1)}, \dots, z_j^{(\ell+m)})^T \in \mathbb{Z}^{\ell+m}$ for all $j \in \{1, \dots, \ell\}$ and let $\bar{x}_i = (x_i^{(1)}, \dots, x_i^{(\ell+m)})^T \in \mathbb{Z}^{\ell+m}$ for all $i \in \{0, 1, \dots, m\}$. Note that $\{\bar{z}_1, \dots, \bar{z}_\ell, \bar{x}_1 - \bar{x}_0, \dots, \bar{x}_m - \bar{x}_0\}$ is linearly independent. Let $B_2 = (\bar{z}_1, \dots, \bar{z}_\ell, \bar{x}_1 - \bar{x}_0, \dots, \bar{x}_m - \bar{x}_0)$. Then, B_2 is an $(\ell + m) \times (\ell + m)$ matrix. Since

$$B_2 \begin{pmatrix} \theta^{(1)} \\ \vdots \\ \theta^{(\ell)} \\ -\kappa^{(1)} \\ \vdots \\ -\kappa^{(m)} \end{pmatrix} = \bar{x}_0,$$

from Cramer's rule, it follows that for all $j \in \{1, \dots, \ell\}$,

$$\theta^{(j)} = \frac{1}{\det B_2} \cdot \det(\bar{z}_1, \dots, \bar{z}_{j-1}, \bar{x}_0, \bar{z}_{j+1}, \dots, \bar{z}_\ell, \bar{x}_1 - \bar{x}_0, \dots, \bar{x}_m - \bar{x}_0) \in \mathbb{Q},$$

and for all $i \in \{1, \dots, m\}$,

$$\kappa^{(i)} = -\frac{1}{\det B_2} \cdot \det(\bar{z}_1, \dots, \bar{z}_\ell, \bar{x}_1 - \bar{x}_0, \dots, \bar{x}_{i-1} - \bar{x}_0, \bar{x}_0, \bar{x}_{i+1} - \bar{x}_0, \dots, \bar{x}_m - \bar{x}_0) \in \mathbb{Q}_{++}.$$

We have

$$|\det B_2| \theta^{(j)} \in \mathbb{Z} \quad \text{for all } j \in \{1, \dots, \ell\}, \text{ and}$$

$$|\det B_2| \kappa^{(i)} \in \mathbb{Z}_{++} \quad \text{for all } i \in \{1, \dots, m\}.$$

Therefore, from $\kappa^{(0)} > 0$ and $\kappa^{(0)} = 1 - \sum_{j=1}^m \kappa^{(j)}$, it follows that $|\det B_2| \kappa^{(0)} \in \mathbb{Z}_{++}$.

Using Hadamard's inequality (Lemma 4), we also have

$$1 \leq |\det B_2| \leq (\ell + m)^{\frac{\ell+m}{2}} K^\ell (2N)^m$$

and for all $j \in \{1, \dots, \ell\}$,

$$\begin{aligned} |\det B_2| |\theta^{(j)}| &= |\det(\bar{z}_1, \dots, \bar{z}_{j-1}, \bar{x}_0, \bar{z}_{j+1}, \dots, \bar{z}_\ell, \bar{x}_1 - \bar{x}_0, \dots, \bar{x}_m - \bar{x}_0)| \\ &\leq (\ell + m)^{\frac{\ell+m}{2}} K^{\ell-1} N (2N)^m. \end{aligned}$$

In a manner similar to the steps of Case 1, we can show that there exists a vector $(\pi^{(1)}, \dots, \pi^{(s)}) \in \mathbb{Q}_+^s$ such that

$$\begin{aligned} & |\det B_2| \pi^{(j)} \in \mathbb{Z}_+ \quad \text{for all } j \in \{1, \dots, s\}, \\ & |\det B_2| \left(1 + \sum_{i=1}^s \pi^{(i)} \right) \\ & \leq (\ell + m)^{\frac{\ell+m}{2}} K^\ell (2N)^m + \ell (\ell + m)^{\frac{\ell+m}{2}} K^{\ell-1} N (2N)^m (\mu(L, K) - 1), \text{ and} \\ 0 &= \sum_{i=0}^m \frac{\kappa^{(i)}}{1 + \sum_{j=1}^s \pi^{(j)}} x_i + \sum_{i=1}^s \frac{\pi^{(i)}}{1 + \sum_{j=1}^s \pi^{(j)}} z_i. \end{aligned}$$

Let

$$\begin{aligned} q_2 &= |\det B_2| \left(1 + \sum_{i=1}^s \pi^{(i)} \right) \in \mathbb{Z}_{++}, \\ \alpha^{(i)} &= \frac{\kappa^{(i)}}{1 + \sum_{j=1}^s \pi^{(j)}} \in \mathbb{Q}_{++} \quad \text{for all } i \in \{0, 1, \dots, m\}, \text{ and} \\ \beta^{(i)} &= \frac{\pi^{(i)}}{1 + \sum_{j=1}^s \pi^{(j)}} \in \mathbb{Q}_+ \quad \text{for all } i \in \{1, \dots, s\}. \end{aligned}$$

Then, we have

$$\begin{aligned} q_2 &\leq q, \\ \sum_{i=0}^m \alpha^{(i)} + \sum_{j=1}^s \beta^{(j)} &= 1, \\ q_2 \alpha^{(i)} &\in \mathbb{Z}_{++} \quad \text{for all } i \in \{0, 1, \dots, m\}, \\ q_2 \beta^{(j)} &\in \mathbb{Z}_+ \quad \text{for all } j \in \{1, \dots, s\}, \text{ and} \\ \sum_{i=0}^m \alpha^{(i)} x_i + \sum_{j=1}^s \beta^{(j)} z_j &= 0. \end{aligned}$$

Therefore, we obtain the required assertion when Case 2 occurs. This completes the proof of Lemma 5. ■

Lemma 6 *Let $v \in \mathbb{Z}^L$ and let E be a nonempty subset of $\{x \in \mathbb{Z}^L \mid x \geq v\}$ satisfying $E + \mathbb{Z}_+^L = E$. Let H be a linear subspace spanned by some elements of \mathbb{Z}^L . If $\text{co}(E) \cap H = \emptyset$, then there exists a vector $p \in H^\perp \cap \mathbb{Z}_+^L$ and a positive real number ε such that for all $z \in \text{co}(E)$,*

$$p \cdot z \geq \varepsilon.$$

Proof. This can be easily obtained from Theorem 5.2 of Inoue [8]. ■

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