Indivisible commodities and the nonemptiness of the weak core

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Abstract

We consider a sufficient condition for the nonemptiness of the weak core in a finite exchange economy where every commodity is available only in integer quantities. We show that if the aggregate upper contour set is discretely convex, then the weak core is nonempty. In addition, we give two sufficient conditions for the aggregate upper contour set to be discretely convex. One is that every upper contour set of every agent is $M^\natural$-convex. The other is that the number of commodities is two and every agent's preference relation is weakly monotone and discretely convex.

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1 Introduction

Indivisibility of commodities can make the weak core empty. Shapley and Scarf (1974) gave an example of an economy with three agents and nine indivisible commodities, the weak core of which was empty. In this example, agents' utility functions are concave. Thus, the emptiness of the weak core came only from the indivisibility of the commodities.

In our economy, every commodity is available only in integer quantities. Agents can consume multiple types of commodities and multiple units of every commodity. We consider the nonemptiness of the weak core of such an economy. When the number of

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agents is one or two, the weak core coincides with the set of individually rational weakly Pareto-efficient allocations. This special structure enables us to show the nonemptiness of the weak core without relying on Scarf's (1967) theorem.

Even in an economy with more than two agents, if the sum of the agents' upper contour sets is discretely convex,[1] then the non-transferable utility (NTU) game derived from the economy is balanced and therefore, from Scarf's (1967) theorem, its weak core is nonempty. The discrete convexity of the aggregate upper contour set is not a condition on every agent's preference relation. As the discrete convexity adopted in this paper is not closed under summation, the discrete convexity of every agent's upper contour set is not sufficient for the discrete convexity of the aggregate upper contour set. Our interest is in the sufficient conditions on every agent's preference relation for the aggregate upper contour set to be discretely convex.

We give two sufficient conditions. The first sufficient condition for the discrete convexity of the aggregate upper contour set is that every upper contour set of every agent is $\mathrm{M}^\natural$-convex.[2] In an economy with only one commodity, if agents' preference relations are weakly monotone, then every upper contour set is $\mathrm{M}^\natural$-convex, and therefore, the weak core is nonempty. In addition, every upper contour set in a house-swapping market game provided by Shapley and Scarf (1974) is $\mathrm{M}^\natural$-convex. Thus, the first sufficient condition explains why every house-swapping market game has a nonempty weak core.

The second sufficient condition is that the number of commodities is two and every agent's preference relation is weakly monotone and discretely convex. As mentioned earlier, if the number of agents is one or two, the weak core is nonempty. Therefore, under some assumptions, if either the number of agents or the number of commodities is one or two, the weak core is nonempty. This result cannot be extended to economies with more than two agents and more than two commodities. We give an example (Example 3 in Section 6) of an economy with three agents and three commodities such that every agent's preference relation is weakly monotone and discretely convex, but the weak core is empty.

In other work, Danilov et al. (2001) gave an equilibrium existence theorem in an economy with many indivisible commodities and one perfectly divisible commodity. They showed that if the sum of the discrete parts of agents' demand sets is discretely convex, then an equilibrium exists. Therefore, they focused on the same property that we do in this paper. In addition, they showed why an equilibrium exists in Gale's (1984) model. In Gale's (1984) model, the discrete part of every agent's demand set forms a special type of $\mathrm{M}^\natural$-convex set. Actually, the same $\mathrm{M}^\natural$-convex set plays an essential role in Shapley and Scarf's (1974) house-swapping market game.

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[1] The precise definition of discrete convexity is given in Section 3.

[2] The precise definition of $\mathrm{M}^\natural$-convexity is given in Section 4.

Inoue (2006) considered the second fundamental theorem of welfare economics in an economy with indivisible commodities. (The commodity space of his economy is the same as ours.) He showed that if the number of commodities is one or two and if every agent's preference relation is weakly monotone and discretely convex, then every Pareto-efficient allocation can be supported as an equilibrium. It should be emphasized that the assumptions of Inoue's (2006) theorem coincide with our second sufficient condition for the discrete convexity of the aggregate upper contour set.

This paper is organized as follows. In Section 2, we give the precise description of our economy. In Section 3, we state the relationship between the discrete convexity of the aggregate upper contour set and the nonemptiness of the weak core. In Section 4, we gather some properties of discretely convex sets. In Section 5, we give two sufficient conditions for the aggregate upper contour set to be discretely convex. In Section 6, we give an example of an economy with three agents and three commodities, the weak core of which is empty. In Section 7, we show that in a house-swapping market game, agents' preference relations satisfy our first sufficient condition for the discrete convexity of the aggregate upper contour set. In the Appendixes, we give the proofs of results stated in the previous sections.

2 Model

We begin with some notation. Let $\mathbb{R}$ and $\mathbb{Z}$ be the sets of real numbers and integers, respectively. For a finite set A, an element of $\mathbb{R}^A$ is denoted by $v = (v^{(a)})_{a \in A}$, whereas for a natural number n, an element of $\mathbb{R}^n$ is denoted by $v = (v^{(1)}, \dots, v^{(n)})$. For $v, w \in \mathbb{R}^n$, we write $v \geq w$ if $v^{(i)} \geq w^{(i)}$ for every $i \in \{1, \dots, n\}$. The symbol 0 denotes the origin in $\mathbb{R}^n$, as well as the real number zero. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geq 0\}$, and $\mathbb{Z}^n_+ = \{z \in \mathbb{Z}^n \mid z \geq 0\}$. For a subset C of $\mathbb{R}^n$, its convex hull is denoted by $\mathrm{co}(C)$.

We consider a finite exchange economy with L commodities, where L is a natural number. Every commodity is available only in integer quantities. Agents can consume multiple types of commodities and can consume multiple units of every commodity. Thus, the commodity space of our economy is given by $\mathbb{Z}^L$. Let A be a finite set of agents. For simplicity, we assume that every agent has the same consumption set $\mathbb{Z}^L_+$. An agent $a \in A$ is characterized by utility function $u_a : \mathbb{Z}^L_+ \to \mathbb{R}$ and endowment vector $e_a \in \mathbb{Z}^L_+$. Our analysis could begin from preference relations instead of utility functions. As the consumption set $\mathbb{Z}^L_+$ is countable, every reflexive, transitive, and complete preference relation can be represented by a utility function. Therefore, we can use utility functions and preference relations interchangeably. An economy $\mathcal{E}$ is given by a list of the commodity space and agents' characteristics, i.e., $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a \in A})$.

Given an economy $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a \in A})$, we use $\mathcal{A}$ to denote the set of all coalitions, i.e., $\mathcal{A} = 2^A \setminus \{\emptyset\}$. For a coalition $S \in \mathcal{A}$, we use $F(S)$ to denote the set of feasible allocations within the coalition S, i.e.,

$$F(S) = \left\{ (x_a)_{a \in S} \in \left(\mathbb{Z}_+^L\right)^S \,\middle|\, \sum_{a \in S} x_a = \sum_{a \in S} e_a \right\}.$$

A feasible allocation within A is simply called a feasible allocation. Note that for every $S \in \mathcal{A}$, $F(S)$ is a finite set.

The main concept we focus on is the weak core. Its precise definition is as follows.

Definition 1 A feasible allocation $(x_a)_{a \in A}$ for economy $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a \in A})$ is called a *weak core allocation* if there exists no $S \in \mathcal{A}$ and $(y_a)_{a \in S} \in F(S)$ such that for any $a \in S$,

$$u_a(y_a) > u_a(x_a).$$

The set of all weak core allocations for $\mathcal{E}$ is called the *weak core* of $\mathcal{E}$ and is denoted by $C_{\mathrm{W}}(\mathcal{E})$.

In the literature, the weak core is simply called the core, but, because of indivisibility of commodities, the size of cores depends heavily on improvement defining the cores. To distinguish this notion of the core from other notions, we refer to it as the weak core.[3]

Scarf (1967) gave a sufficient condition for the nonemptiness of the weak core of NTU games. An NTU game is a correspondence V of $\mathcal{A}$ into $\mathbb{R}^A$; V assigns to every coalition S its possible utility allocations $V(S) \subset \mathbb{R}^A$. The weak core $C_{\mathrm{W}}(V)$ of NTU game V is defined by

$$C_{\mathrm{W}}(V) = \{v \in V(A) \mid \text{there exists no } S \in \mathcal{A} \text{ and } w \in V(S) \text{ such that for any } a \in S,\ w^{(a)} > v^{(a)}\}.$$

A subclass $\mathcal{B}$ of $\mathcal{A}$ is *balanced* if there are nonnegative coefficients $(\lambda_S)_{S \in \mathcal{B}}$ such that for every $a \in A$, $\sum_{S \in \mathcal{B}} \lambda_S \chi_S^{(a)} = 1$, where

$$\chi_S^{(a)} = \begin{cases} 1 & \text{if } a \in S, \\ 0 & \text{if } a \notin S. \end{cases}$$

The numbers $(\lambda_S)_{S \in \mathcal{B}}$ are called *balancing coefficients for* $\mathcal{B}$. An NTU game V of $\mathcal{A}$ into $\mathbb{R}^A$ is *balanced* if for every balanced family $\mathcal{B}$ of $\mathcal{A}$, $\bigcap_{S \in \mathcal{B}} V(S) \subset V(A)$.

Scarf (1967) showed that every balanced NTU game V that satisfies the following conditions (a)-(c) has a nonempty weak core.

(a) For every $S \in \mathcal{A}$, $V(S)$ is closed.

(b) For every $S \in \mathcal{A}$, if $v \in V(S), w \in \mathbb{R}^A$, and for every $a \in S$, $v^{(a)} \geq w^{(a)}$, then $w \in V(S)$.

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[3] For other competing notions of cores, see Inoue (2005).

(c) For every $S \in \mathcal{A}$, the set $\left[V(S) \setminus \bigcup_{a \in S} \operatorname{int} V(\{a\})\right] \cap (\mathbb{R}^S \times \{0\}^{A \setminus S})$ is nonempty and bounded, where $\operatorname{int} V(\{a\})$ refers to the interior of $V(\{a\})$.

The NTU game $V_{\mathcal{E}}$ derived from economy $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a \in A})$ is defined by, for every $S \in \mathcal{A}$,

$$V_{\mathcal{E}}(S) = \left\{ \left(v^{(a)}\right)_{a \in A} \in \mathbb{R}^A \;\middle|\; \text{there exists } (x_a)_{a \in S} \in F(S) \text{ such that for every } a \in S,\ u_a(x_a) \geq v^{(a)} \right\}.$$

It can be shown that every NTU game $V_{\mathcal{E}}$ derived from an economy $\mathcal{E}$ satisfies conditions (a)-(c). In addition, $C_{\mathrm{W}}(\mathcal{E}) \neq \emptyset$ if and only if $C_{\mathrm{W}}(V_{\mathcal{E}}) \neq \emptyset$. Therefore, by virtue of Scarf's (1967) theorem, if $V_{\mathcal{E}}$ is balanced, then the weak core $C_{\mathrm{W}}(\mathcal{E})$ of $\mathcal{E}$ is nonempty.

Our first result is that every economy with one or two agents has a nonempty weak core.

Theorem 1 *Let $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a \in A})$ be an economy with $1 \leq \#A \leq 2$. Then, the weak core $C_{\mathrm{W}}(\mathcal{E})$ of $\mathcal{E}$ is nonempty.*

Proof. The assertion is clear if $\#A = 1$, because $F(A)$ is a singleton. When $\#A = 2$, we give two proofs. The first proof relies on Scarf's (1967) theorem. The method of the second proof was used by Emmerson (1972, Theorem 1) when he showed the existence of a Pareto-efficient allocation in the discrete commodity space. Let $A = \{1, 2\}$.
First proof. It suffices to show that $V_{\mathcal{E}}$ is balanced. Note that balanced families of $\mathcal{A}$ are limited essentially to $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$. We can easily obtain $V_{\mathcal{E}}(\{1\}) \cap V_{\mathcal{E}}(\{2\}) \subset V_{\mathcal{E}}(\{1, 2\})$, so $V_{\mathcal{E}}$ is balanced. Therefore, $C_{\mathrm{W}}(\mathcal{E}) \neq \emptyset$.
Second proof. Suppose that $C_{\mathrm{W}}(\mathcal{E}) = \emptyset$. Then, for every $(x_1, x_2) \in F(A)$ with $u_1(x_1) \geq u_1(e_1)$ and $u_2(x_2) \geq u_2(e_2)$, there exists $(y_1, y_2) \in F(A)$ such that $u_1(y_1) > u_1(x_1)$ and $u_2(y_2) > u_2(x_2)$. Therefore, we can construct an infinite sequence $(x_1^n, x_2^n)_n$ of $F(A)$ such that

$$\begin{aligned} &u_1(e_1) < u_1(x_1^1) < u_1(x_1^2) < \cdots < u_1(x_1^n) < u_1(x_1^{n+1}) < \cdots, \quad \text{and} \\ &u_2(e_2) < u_2(x_2^1) < u_2(x_2^2) < \cdots < u_2(x_2^n) < u_2(x_2^{n+1}) < \cdots. \end{aligned}$$

As the set $F(A)$ is finite, there are n and m with $n < m$ such that $(x_1^n, x_2^n) = (x_1^m, x_2^m)$. This contradicts $u_1(x_1^n) < u_1(x_1^m)$ and $u_2(x_2^n) < u_2(x_2^m)$. Hence, $C_{\mathrm{W}}(\mathcal{E}) \neq \emptyset$. ■

3 Discrete convexity and the nonemptiness of the weak core

In an economy with more than two agents, the discrete convexity of the aggregate upper contour set plays an important role for the nonemptiness of the weak core. The precise definition of discrete convexity is as follows.[4]

Definition 2 A subset S of $\mathbb{Z}^L$ is *discretely convex* if $\mathrm{co}(S) \cap \mathbb{Z}^L = S$ holds.

Note that, for every convex subset S of $\mathbb{R}^L$, the set $S \cap \mathbb{Z}^L$ is discretely convex. It should be emphasized that the sum of two discretely convex sets is not always discretely convex. We give a well-known example. (See also Example 2 in Section 4.)

Example 1 Let $S_1 = \{(0,1),(1,0)\}$ and $S_2 = \{(0,0),(1,1)\}$. Then, S_1 and S_2 are both discretely convex. As $S_1 + S_2 = \{(0,1),(1,0),(1,2),(2,1)\}$, it follows that $(1,1) \in \mathrm{co}(S_1 + S_2) \cap \mathbb{Z}^2$ but $(1,1) \notin S_1 + S_2$. Hence, $S_1 + S_2$ is not discretely convex.

The properties of discretely convex sets are gathered in Section 4.

Concerning the nonemptiness of the weak core, the next theorem clarifies the difference between an economy with perfectly divisible commodities and our economy with indivisible commodities. In an economy with perfectly divisible commodities, if every agent has a convex preference relation, then the aggregate upper contour set is convex because the sum of two convex sets is always convex. On the other hand, in our economy, as discussed in Example 1, the discrete convexity is not closed under summation. Therefore, the discrete convexity of every agent's upper contour set is not sufficient for the discrete convexity of the aggregate upper contour set. In the following theorem, we directly assume the discrete convexity of the aggregate upper contour set.

The method of proof of the following theorem is essentially the same as Scarf's (1967, Section 2), but we give the proof in Appendix A in order to clarify the role of the discrete convexity of the aggregate upper contour set.

Theorem 2 *Let* $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a \in A})$ *be an economy. If for every* $(x_a)_{a \in A} \in (\mathbb{Z}^L_+)^A$*, the aggregate upper contour set* $\sum_{a \in A} \{y \in \mathbb{Z}^L_+ \mid u_a(y) \geq u_a(x_a)\}$ *is discretely convex, then the NTU game* $V_{\mathcal{E}}$ *is balanced, and therefore, the weak core* $C_{\mathrm{W}}(\mathcal{E})$ *of* $\mathcal{E}$ *is nonempty.*

Theorem 2 is mathematically general, but the discrete convexity of the aggregate upper contour set is not a condition on every agent's preference relation. In Section 5, we give two sufficient conditions on every agent's preference relation for the aggregate upper contour set to be discretely convex.

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[4] The term discrete convexity follows Emmerson (1972). Danilov et al. (2001) and Danilov and Koshevoy (2004) referred to the property $\mathrm{co}(S) \cap \mathbb{Z}^L = S$ as *pseudo-convexity*, and Murota (2003) referred to it as *hole-freeness*.

4 Properties of discretely convex sets

In this section, we sum up the properties of discretely convex sets. One of the classes of discretely convex subsets of $\mathbb{Z}^L$, such that the sum of two sets belongs to the class itself, is the class of M$^\natural$-convex sets.[5] For every $i \in \{1, \ldots, L\}$, let $\chi_i \in \mathbb{Z}^L$ be the ith unit vector, i.e., $\chi_i^{(i)} = 1$ and $\chi_i^{(j)} = 0$ if $i \neq j$.

Definition 3 A subset S of $\mathbb{Z}^L$ is *M$^\natural$-convex* if, for every $x, y \in S$ and every $i \in \{1, \ldots, L\}$ with $x^{(i)} > y^{(i)}$, either (i) or (ii) holds.

(i) $x - \chi_i \in S$ and $y + \chi_i \in S$.

(ii) There exists $j \in \{1, \ldots, L\}$ with $x^{(j)} < y^{(j)}$ such that $x - \chi_i + \chi_j \in S$ and $y + \chi_i - \chi_j \in S$.

M$^\natural$-convex sets have the following properties.

(M$^\natural$ 1) Every M$^\natural$-convex set is discretely convex.

(M$^\natural$ 2) The sum of two M$^\natural$-convex sets is M$^\natural$-convex.

(M$^\natural$ 3) The set $\mathbb{Z}_+^L$ is M$^\natural$-convex.

(M$^\natural$ 4) Every subset of $\{0, \chi_1, \ldots, \chi_L\}$ is M$^\natural$-convex.

The class of M$^\natural$-convex sets is appealing because of these properties. As $\mathbb{Z}_+^L$ is M$^\natural$-convex, upper contour sets of some weakly monotone preference relations are M$^\natural$-convex. In addition, a subset of $\{0, \chi_1, \ldots, \chi_L\}$ is a simple M$^\natural$-convex set, and this simple M$^\natural$-convex set makes the weak core in a house-swapping market game nonempty (see Theorem 6 in Section 7).

An M$^\natural$-convex set in $\mathbb{Z}^L$ is also characterized as the projection of an M-convex set in $\mathbb{Z}^{L+1}$ along a coordinate axis.[6] The equivalence between this characterization by the projection and the above definition was shown by Murota and Shioura (1999). Murota (2003, Chapter 4) showed that (M1) every M-convex set is discretely convex, and (M2) the sum of two M-convex sets is M-convex. From (M1) and (M2), by using the characterization of M$^\natural$-convexity by the projection of M-convex set, we can easily obtain the properties (M$^\natural$ 1) and (M$^\natural$ 2). The properties (M$^\natural$ 3) and (M$^\natural$ 4) follow from the definition.

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[5] An M$^\natural$-convex set coincides with the set of integral vectors of a so-called integral generalized polymatroid. The term M$^\natural$-convexity, which should be read M-natural-convexity, follows Murota (2003). Danilov et al. (2001) and Danilov and Koshevoy (2004) used the term PM-set in the same sense as M$^\natural$-convex set.

[6] A subset S of $\mathbb{Z}^L$ is *M-convex* if for every $x, y \in S$ and every $i \in \{1, \ldots, L\}$ with $x^{(i)} > y^{(i)}$, there exists $j \in \{1, \ldots, L\}$ with $x^{(j)} < y^{(j)}$ such that $x - \chi_i + \chi_j \in S$ and $y + \chi_i - \chi_j \in S$. Therefore, every M-convex set is M$^\natural$-convex, and every intersection of an M$^\natural$-convex set in $\mathbb{Z}^L$ and a hyperplane $\{x \in \mathbb{R}^L \mid \sum_{i=1}^L x^{(i)} = r\}$ with $r \in \mathbb{Z}$ is M-convex.

Remark 1 The class of M$^\natural$-convex sets is not maximal of all classes that satisfy the conditions that every set in the class is discretely convex and the sum of two sets in the class belongs to the class itself. For example, in $\mathbb{Z}^2$,

$$\mathcal{S} = \{S \subset \mathbb{Z}^2 \,|\, S \text{ is M}^\natural\text{-convex}\} \cup \{S \subset \mathbb{Z}^2 \,|\, S \text{ is discretely convex and } S + \mathbb{Z}^2_+ = S\}$$

is a strictly larger class than the class of M$^\natural$-convex sets, and $\mathcal{S}$ is closed under summation. Indeed, if $S_1 \subset \mathbb{Z}^2$ is M$^\natural$-convex and if $S_2 \subset \mathbb{Z}^2$ is discretely convex and $S_2 + \mathbb{Z}^2_+ = S_2$, then $S_1 + S_2 = (S_1 + \mathbb{Z}^2_+) + (S_2 + \mathbb{Z}^2_+)$. From (M$^\natural$ 1)-(M$^\natural$ 3), $S_1 + \mathbb{Z}^2_+$ is discretely convex, and therefore, from Theorem 3 below, $S_1 + S_2$ is discretely convex.

The class $\mathcal{S}$ is not closed under taking faces. For example, $\mathrm{co}(\{(0,2),(3,0)\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$ is in $\mathcal{S}$, but the set $\{(0,2),(3,0)\}$ of integral points of the face $\mathrm{co}(\{(0,2),(3,0)\})$ is not in $\mathcal{S}$. Danilov and Koshevoy (2004) showed that the class of M$^\natural$-convex sets is a maximal class of discretely convex sets of all classes that are closed under summation, under taking faces, and under some operations.

Next, we consider one-dimensional discretely convex sets. For $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z} \cup \{+\infty\}$, the set $\{x \in \mathbb{Z} \,|\, a \leq x \leq b\}$ is called an *integral interval.*

Remark 2 For a subset S of $\mathbb{Z}$, the following three conditions are equivalent.

1. The set S is discretely convex.
2. The set S is an integral interval.
3. The set S is M$^\natural$-convex.

The following theorem gives us a sufficient condition for the sum of discretely convex sets in $\mathbb{Z}^2$ to be discretely convex. The proof is given in Appendix B.

Theorem 3 *Let S_1 and S_2 be nonempty discretely convex subsets of $\mathbb{Z}^2$. If $S_i + \mathbb{Z}^2_+ = S_i$ for every $i \in \{1,2\}$, then $S_1 + S_2$ is discretely convex.*

In Theorem 3, the condition that $S_i + \mathbb{Z}^2_+ = S_i$ for every $i \in \{1,2\}$ is indispensable (recall Example 1). It is not sufficient that only one set satisfies the condition $S_i + \mathbb{Z}^2_+ = S_i$. The next example illustrates this point.

Example 2 Let $S_1 = \{(0,2),(3,0)\}$ and $S_2 = \mathbb{Z}^2_+$. Then, both S_1 and S_2 are discretely convex and $S_1 + \mathbb{Z}^2_+ \supsetneq S_1$. We have $(2,1) \in \mathrm{co}(S_1 + S_2) \cap \mathbb{Z}^2$ but $(2,1) \notin S_1 + S_2$. Therefore, $\mathrm{co}(S_1 + S_2) \cap \mathbb{Z}^2 \supsetneq S_1 + S_2$.

In addition, Theorem 3 relies on the two-dimensionality. If the dimension L of sets S_1 and S_2 is higher than two, then $S_1 + S_2$ is not always discretely convex, even though both

S_1 and S_2 are discretely convex and for every $i \in \{1, 2\}$, $S_i + \mathbb{Z}^L_+ = S_i$ holds. Example 3 in Section 6 illustrates this point.

The next property of discretely convex sets holds in any dimension. It can be easily shown.

(D1) Let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of discretely convex sets in $\mathbb{Z}^L$. If for every $\lambda, \mu \in \Lambda$, either $S_\lambda \subset S_\mu$ or $S_\mu \subset S_\lambda$ holds, then $\bigcup_{\lambda\in\Lambda} S_\lambda$ is discretely convex.

5 Nonemptiness of the weak core

In this section, we give two sufficient conditions for the nonemptiness of the weak core. The first sufficient condition is that every upper contour set of every agent is M$^\natural$-convex.

Theorem 4 *Let* $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a\in A})$ *be an economy. If for every* $a \in A$ *and every* $x_a \in \mathbb{Z}^L_+$*, the upper contour set* $\{y \in \mathbb{Z}^L_+ \mid u_a(y) \geq u_a(x_a)\}$ *is M$^\natural$-convex, then the weak core* $C_{\mathrm{W}}(\mathcal{E})$ *of* $\mathcal{E}$ *is nonempty.*

Proof. By virtue of Theorem 2 in Section 3, it suffices to show the discrete convexity of the aggregate upper contour set, but this follows from properties (M$^\natural$ 1) and (M$^\natural$ 2). ■

The nonemptiness of the weak core of a house-swapping market game provided by Shapley and Scarf (1974) follows from this theorem (see Theorem 6 in Section 7).

From Remark 2 in Section 4, if there is only one commodity in the economy, we have the following as a corollary.

Corollary 1 *Let* $\mathcal{E} = (\mathbb{Z}, (u_a, e_a)_{a\in A})$ *be an economy with only one commodity. If for every* $a \in A$ *and every* $x_a \in \mathbb{Z}_+$*, the upper contour set* $\{y \in \mathbb{Z}_+ \mid u_a(y) \geq u_a(x_a)\}$ *is an integral interval, then the weak core* $C_{\mathrm{W}}(\mathcal{E})$ *of* $\mathcal{E}$ *is nonempty.*

Note that if every agent's preference relation is weakly monotone, i.e., $[\, x, y \in \mathbb{Z}_+,\ x \leq y\,]$ implies $u_a(x) \leq u_a(y)$, then every upper contour set is an integral interval.

The second sufficient condition is that the number of commodities is two and every agent's preference relation is weakly monotone and discretely convex.

Theorem 5 *Let* $\mathcal{E} = (\mathbb{Z}^2, (u_a, e_a)_{a\in A})$ *be an economy with two commodities satisfying the following conditions.*

(i) [weak monotonicity of preference relations]
For every $a \in A$, $u_a : \mathbb{Z}^L_+ \to \mathbb{R}$ *is weakly monotone, i.e., if* $x, y \in \mathbb{Z}^L_+$ *and* $x \leq y$*, then* $u_a(x) \leq u_a(y)$*.*

(ii) [discrete convexity of agents' upper contour sets]
For every $a \in A$ *and every* $x_a \in \mathbb{Z}^L_+$, *the upper contour set* $\{y \in \mathbb{Z}^L_+ \mid u_a(y) \geq u_a(x_a)\}$ *is discretely convex.*

Then, the weak core $C_{\mathrm{W}}(\mathcal{E})$ *of* $\mathcal{E}$ *is nonempty.*

Proof. By virtue of Theorem 2 in Section 3, it suffices to show the discrete convexity of the aggregate upper contour set, but this is a direct consequence of Theorem 3 in Section 4. ■

From the argument in Remark 1 in Section 4, Theorem 5 can be strengthened as follows. In an economy with two commodities, if some agents' preference relations are weakly monotone and discretely convex and if other agents' preference relations are M$^\natural$-convex, then the weak core is nonempty.

6 Example of an economy with an empty weak core

As was shown in Theorems 1 and 5, and Corollary 1, if either the number of agents or the number of commodities is one or two and if every agent's preference relation is weakly monotone and discretely convex, then the weak core is nonempty. In an economy where both numbers are greater than two, the weak core can be empty. Shapley and Scarf (1974, Section 8) gave an example of an economy with three agents and nine commodities, the weak core of which was empty. Konishi et al. (2001, Example 5.1) gave an example of an economy with four agents and five commodities, the weak core of which was empty.[7] In both examples, agents can consume at most one unit of every commodity because there exists only one unit of every commodity in the economy.

We give an example of an economy with three agents and three commodities, the weak core of which is empty. In light of Theorems 1 and 5, and Corollary 1, an economy with three agents and three commodities is the smallest one in the class of economies with an empty weak core. The emptiness of the weak core stems from that discrete convexity is not closed under summation if the dimension is higher than two. Therefore, for every $L \geq 3$ and every $\#A \geq 3$, there could exist an economy with L commodities and $\#A$ agents such that every agent's preference relation is weakly monotone and discretely convex but the weak core is empty.

Example 3 Let $A = \{1, 2, 3\}$ be the set of agents. Let $L = 3$. Then, every agent's consumption set is $\mathbb{Z}^3_+$. Agents' endowment vectors are given by

$$e_1 = (0, 0, 1),$$

---

[7] Konishi et al. (2001) assumed the additive separability of utility functions. Thus, agents' preference relations have no complementarity among commodities.

$$
\begin{aligned}
e_2 &= (1,1,0), \text{and}\\
e_3 &= (1,1,0).
\end{aligned}
$$

Agents' utility functions are given by

$$
\begin{aligned}
u_1(x) &= \min\{1,(1,0,0)\cdot x\}+\min\{1,(1,1,0)\cdot x\}+\min\{1,(1,1,1)\cdot x\}-1,^{8}\\
u_2(x) &= \min\{1,(1,0,0)\cdot x-1\}+2\min\{1,(1,2,0)\cdot x-1,(1,0,2)\cdot x-1\}\\
&\quad +\min\{1,(1,2,0)\cdot x-1,(1,0,1)\cdot x\}-1, \text{ and}\\
u_3(x) &= \min\{1,(1,0,0)\cdot x,(0,0,1)\cdot x\}+\min\{1,(2,1,0)\cdot x-1,(0,1,2)\cdot x-1\}\\
&\quad +\min\{1,(2,1,0)\cdot x-1,(0,1,1)\cdot x\}-1.
\end{aligned}
$$

In figure i, the number in the square bracket refers to agent i's utility level. Note that every u_i is a concave function on $\mathbb{R}^3$, although agents' utilities are defined only on their consumption set $\mathbb{Z}^3_+$. Therefore, every upper contour set of every agent is discretely convex because $S \cap \mathbb{Z}^3$ is discretely convex whenever $S \subset \mathbb{R}^3$ is convex. Note also that every utility function is weakly monotone. In contrast to Konishi et al. (2001), the preference relations of agents 2 and 3 have complementarity among commodities, i.e., for $i \in \{2,3\}$, the preference relation of agent i cannot be represented by a utility function u_i of the form $u_i(x) = \sum_{j=1}^{3} w_i^j(x^{(j)})$, where every satisfaction function $w_i^j : \mathbb{R}_+ \to \mathbb{R}$ is concave.[9] We call this economy $\mathcal{E}$.

In order to show that the weak core $C_{\mathrm{W}}(\mathcal{E})$ of $\mathcal{E}$ is empty, we shall find corners of $V_{\mathcal{E}}(S)$.[10] It is clear that

$$
\begin{aligned}
V_{\mathcal{E}}(\{1\}) &: (0,-,-),\\
V_{\mathcal{E}}(\{2\}) &: (-,0,-),\\
V_{\mathcal{E}}(\{3\}) &: (-,-,0),
\end{aligned}
$$

where, for example, "$V_{\mathcal{E}}(\{1\}) : (0,-,-)$" means $V_{\mathcal{E}}(\{1\}) = \{v \in \mathbb{R}^3 \mid v^{(1)} \le 0\}$. Therefore, $v \in \mathbb{R}^3$ with $v^{(i)} < 0$ can be improved upon by agent i. Thus, it suffices to consider nonnegative corners.

In addition, as $u_1(0) = -1$, and for every $x \in \mathbb{Z}^3_+$ with $\sum_{j=1}^{3} x^{(j)} \le 1$, $u_2(x) \le -1$ and $u_3(x) \le -1$, it suffices to consider allocations (x_1, x_2, x_3) such that

$$
\sum_{j=1}^{3} x_1^{(j)} = 1, \text{ and } \sum_{j=1}^{3} x_2^{(j)} = \sum_{j=1}^{3} x_3^{(j)} = 2.
$$

---

[8] For $a, b \in \mathbb{R}^3$, $a \cdot b = \sum_{i=1}^{3} a^{(i)} b^{(i)}$.

[9] We show this fact in Appendix C.

[10] A vector $(v^{(a)})_{a \in S}$ is a *corner* of $V_{\mathcal{E}}(S)$ if there exists no $(w^{(a)})_{a \in A} \in V_{\mathcal{E}}(S)$ such that $w^{(a)} \ge v^{(a)}$ for any $a \in S$ and at least one of these inequalities is strict.

We can easily verify that nonnegative corners of $V_{\mathcal{E}}(S)$ with a two-person coalition S are as follows:

$$\begin{aligned} V_{\mathcal{E}}(\{1,2\}) \cap \mathbb{R}^3_+ &\ :\ (2,1,-), \\ V_{\mathcal{E}}(\{2,3\}) \cap \mathbb{R}^3_+ &\ :\ (-,3,1), \\ V_{\mathcal{E}}(\{3,1\}) \cap \mathbb{R}^3_+ &\ :\ (1,-,2), \end{aligned}$$

where, for example, "$V_{\mathcal{E}}(\{1,2\}) \cap \mathbb{R}^3_+ : (2,1,-)$" means $V_{\mathcal{E}}(\{1,2\}) \cap \mathbb{R}^3_+ = \{v \in \mathbb{R}^3_+ \mid v^{(1)} \leq 2 \text{ and } v^{(2)} \leq 1\}$.

We can also verify that nonnegative corners of $V_{\mathcal{E}}(\{1,2,3\})$ are as follows:

$$V_{\mathcal{E}}(\{1,2,3\}) \cap \mathbb{R}^3_+ \ :\ (2,1,0),\ (0,3,1),\ \text{and } (1,0,2).$$

Therefore, utility allocation $(2,1,0)$ can be improved upon by coalition $\{2,3\}$, utility allocation $(0,3,1)$ can be improved upon by coalition $\{3,1\}$, and utility allocation $(1,0,2)$ can be improved upon by coalition $\{1,2\}$. Thus, the weak core of $V_{\mathcal{E}}$ is empty, and therefore, the weak core $C_{\mathrm{W}}(\mathcal{E})$ of $\mathcal{E}$ is empty.

The NTU game $V_{\mathcal{E}}$ is not balanced, and this comes from the failure of the discrete convexity of the aggregate upper contour set. Indeed, $(1,1,1) \in V_{\mathcal{E}}(\{1,2\}) \cap V_{\mathcal{E}}(\{2,3\}) \cap V_{\mathcal{E}}(\{3,1\})$ but $(1,1,1) \notin V_{\mathcal{E}}(\{1,2,3\})$. Thus, $V_{\mathcal{E}}$ is not balanced. We now show that $e_1 + e_2 + e_3 \in \mathrm{co}(\sum_{a \in A}\{x \in \mathbb{Z}^3_+ \mid u_a(x) \geq 1\}) \cap \mathbb{Z}^3$ and $e_1 + e_2 + e_3 \notin \sum_{a \in A}\{x \in \mathbb{Z}^3_+ \mid u_a(x) \geq 1\}$. For every $a \in A$, let $U_a = \{x \in \mathbb{Z}^3_+ \mid u_a(x) \geq 1\}$. Then, we have

$$\begin{aligned} U_1 &= \{(1,0,0),(0,1,0)\} + \mathbb{Z}^3_+, \\ U_2 &= \{(2,0,0),(0,1,1)\} + \mathbb{Z}^3_+, \text{ and} \\ U_3 &= \{(1,0,1),(0,2,0)\} + \mathbb{Z}^3_+.^{11} \end{aligned}$$

The minimal elements of $U_1 + U_2 + U_3$ are as follows:

$$\begin{gathered} (4,0,1),\ (2,1,2),\ (3,1,1),\ (1,2,2), \\ (3,2,0),\ (1,3,1),\ (2,3,0),\ (0,4,1). \end{gathered}$$

In figure 4, circled points indicate minimal elements of $U_1 + U_2 + U_3$, and the boxed point indicates $e_1 + e_2 + e_3 = (2,2,1)$. Therefore, $e_1 + e_2 + e_3 \in \mathrm{co}(U_1 + U_2 + U_3) \cap \mathbb{Z}^3$ but $e_1 + e_2 + e_3 \notin U_1 + U_2 + U_3$. Hence, the aggregate upper contour set $U_1 + U_2 + U_3$ is not discretely convex.

7 House-swapping market game

In this section, we show that every upper contour set in a house-swapping market game is M$^\natural$-convex. Therefore, from Theorem 4 in Section 5, the weak core is nonempty. An

---

[11] We show these equalities in Appendix C.

economy $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a \in A})$ is called a *house-swapping market game* if agents' utility functions satisfy the following conditions:

(U1) For every $a \in A$, every $i \in \{1, \dots, L\}$, and every natural number m, $u_a(m\,\chi_i) = u_a(\chi_i)$.

(U2) For every $a \in A$ and every $i \in \{1, \dots, L\}$, $u_a(\chi_i) > u_a(0)$.

(U3) For every $a \in A$ and every $x \in \mathbb{Z}^L_+$, $u_a(x) = \max\{u_a(x^{(i)}\,\chi_i) \mid i = 1, \dots, L\}$.

Condition (U1) means that consuming several units of every commodity is indifferent to consuming one unit of the commodity. Condition (U2) means that consuming nothing is ranked below all else. Condition (U3) means that consuming several commodities is ranked only equal to the maximum of their separate ranks.

Under these assumptions about agents' utility functions, an interesting situation is that every agent's endowment vector is a unit vector, that is, for every $a \in A$, there exists $i \in \{1, \dots, L\}$ such that $e_a = \chi_i$. Shapley and Scarf (1974) considered a situation where every agent brings one unit of one indivisible commodity to the market and every commodity is differentiated. Thus, in the original paper by Shapley and Scarf (1974), it is assumed that $A = \{1, \dots, L\}$ and $e_i = \chi_i$ for every $i \in A$. As long as we consider the nonemptiness of the weak core, we do not need to make any assumptions about agents' endowment vectors. Here, we simply assume that for every $a \in A$, $e_a \in \mathbb{Z}^L_+$.

Finally, we show that every house-swapping market game has a nonempty weak core.

Theorem 6 *Let $\mathcal{E} = (\mathbb{Z}^L, (u_a, e_a)_{a \in A})$ be a house-swapping market game. Then, for every $a \in A$ and every $x_a \in \mathbb{Z}^L_+$, the upper contour set $\{y \in \mathbb{Z}^L_+ \mid u_a(y) \geq u_a(x_a)\}$ is $M^\natural$-convex. Therefore, the weak core $C_{\mathrm{W}}(\mathcal{E})$ of $\mathcal{E}$ is nonempty.*

Proof. First, we consider the case, as an example, where $L = 3$ and agent a's utility function satisfies

$$u_a(\chi_1) > u_a(\chi_2) > u_a(\chi_3).$$

Agent a's upper contour sets are the following.

- $\{y \in \mathbb{Z}^3_+ \mid u_a(y) \geq u_a(0)\} = \mathbb{Z}^3_+$
- $\{y \in \mathbb{Z}^3_+ \mid u_a(y) \geq u_a(\chi_3)\} = \{\chi_1, \chi_2, \chi_3\} + \mathbb{Z}^3_+$
- $\{y \in \mathbb{Z}^3_+ \mid u_a(y) \geq u_a(\chi_2)\} = \{\chi_1, \chi_2\} + \mathbb{Z}^3_+$
- $\{y \in \mathbb{Z}^3_+ \mid u_a(y) \geq u_a(\chi_1)\} = \{\chi_1\} + \mathbb{Z}^3_+$

In general, agents' upper contour sets are given by the following form:

$$E + \mathbb{Z}^L_+ \quad \text{for some nonempty subset } E \text{ of } \{0, \chi_1, \ldots, \chi_L\}.$$

From properties (M$^\natural$2)-(M$^\natural$4), the sets of this form are M$^\natural$-convex. Therefore, from Theorem 4 in Section 5, the weak core of $\mathcal{E}$ is nonempty. ■

Appendix A: Proof of Theorem 2

Let $\mathcal{B}$ be a balanced family of $\mathcal{A}$ and let $v \in \bigcap_{S\in\mathcal{B}} V_{\mathcal{E}}(S)$. Then, for every $S \in \mathcal{B}$, there exists $(x^S_a)_{a\in S} \in F(S)$ such that for every $a \in S$, $u_a(x^S_a) \geq v^{(a)}$. Let $(\lambda_S)_{S\in\mathcal{B}}$ be the balancing coefficients for $\mathcal{B}$. Then, for every $a \in A$,

$$\sum_{S\in\mathcal{B}} \lambda_S \, \chi^{(a)}_S \, x^S_a \in \mathrm{co}(X_a),$$

where $X_a = \{x \in \mathbb{Z}^L_+ \,|\, u_a(x) \geq v^{(a)}\}$. Therefore, we have

$$\sum_{a\in A}\sum_{S\in\mathcal{B}} \lambda_S \, \chi^{(a)}_S \, x^S_a \in \sum_{a\in A} \mathrm{co}(X_a) = \mathrm{co}\left(\sum_{a\in A} X_a\right).$$

On the other hand,

$$\begin{aligned}
\sum_{a\in A}\sum_{S\in\mathcal{B}} \lambda_S \, \chi^{(a)}_S \, x^S_a &= \sum_{S\in\mathcal{B}} \lambda_S \sum_{a\in A} \chi^{(a)}_S \, x^S_a \\
&= \sum_{S\in\mathcal{B}} \lambda_S \sum_{a\in A} \chi^{(a)}_S \, e_a \\
&= \sum_{a\in A} e_a \sum_{S\in\mathcal{B}} \lambda_S \, \chi^{(a)}_S \\
&= \sum_{a\in A} e_a \in \mathbb{Z}^L.
\end{aligned}$$

Hence, $\sum_{a\in A} e_a \in \mathrm{co}\left(\sum_{a\in A} X_a\right) \cap \mathbb{Z}^L$. If the set $\sum_{a\in A} X_a$ is discretely convex, then we have $v \in V_{\mathcal{E}}(A)$, and therefore, $V_{\mathcal{E}}$ is balanced. Therefore, it remains to show that $\sum_{a\in A} X_a$ is discretely convex.

When the set $\{u_a(x) \,|\, x \in X_a\}$ does not have a minimum, there exists a sequence $(x_{a,n})_n$ of X_a such that $u_a(x_{a,n}) \searrow \inf\{u_a(x) \,|\, x \in X_a\}$ as $n \to \infty$. Let $U_a(x) = \{z \in \mathbb{Z}^L_+ \,|\, u_a(z) \geq u_a(x)\}$. Then, for every n, $U_a(x_{a,n}) \subset U_a(x_{a,n+1})$, and $\bigcup_{n=1}^{\infty} U_a(x_{a,n}) = X_a$. When the set $\{u_a(x) \,|\, x \in X_a\}$ has a minimum, there exists an $x^*_a \in X_a$ with $u_a(x^*_a) = \min\{u_a(x) \,|\, x \in X_a\}$. In this case, for every n, put $x_{a,n} = x^*_a$. Note that for every n, $U_a(x_{a,n}) = U_a(x_{a,n+1})$, and $\bigcup_{n=1}^{\infty} U_a(x_{a,n}) = X_a$. Therefore, in each case, we have $\sum_{a\in A} X_a = \sum_{a\in A} \bigcup_{n=1}^{\infty} U_a(x_{a,n})$. As the equality $\sum_{a\in A} \bigcup_{n=1}^{\infty} U_a(x_{a,n}) = \bigcup_{n=1}^{\infty} \sum_{a\in A} U_a(x_{a,n})$ holds, we have $\sum_{a\in A} X_a = \bigcup_{n=1}^{\infty} \sum_{a\in A} U_a(x_{a,n})$. By assumption, for every n, the set

$\sum_{a\in A} U_a(x_{a,n})$ is discretely convex. In addition, for every $n < m$, $\sum_{a\in A} U_a(x_{a,n}) \subset \sum_{a\in A} U_a(x_{a,m})$. Therefore, from the property (D1) in Section 4, $\bigcup_{n=1}^{\infty} \sum_{a\in A} U_a(x_{a,n})$ is discretely convex. Hence, the set $\sum_{a\in A} X_a$ is discretely convex.

Appendix B: Proof of Theorem 3

For a subset S of $\mathbb{Z}^2$, denote by S' the set of minimal elements of S with respect to $\leq$, i.e.,

$$S' = \{x \in S \,|\, \text{there exists no } y \in S \text{ such that } y \leq x \text{ and } y \neq x\}.$$

Lemma 1 *(Gordan).*[12] *Let S be a nonempty subset of $\mathbb{Z}^2$. If S is bounded from below, i.e., $S \subset \{x\} + \mathbb{Z}^2_+$ for some $x \in \mathbb{Z}^2$, then S' is nonempty and finite.*

For the proof of Lemma 1, see, e.g., Inoue (2005, Lemma 5.1).

Now we are ready for the proof of Theorem 3. First, we suppose that both S_1 and S_2 are bounded from below. For every $i \in \{1, 2\}$, we have

(a) $\emptyset \neq \mathrm{co}(S_i)^e \subset S_i'$, and

(b) $S_i = \mathrm{co}\left(\mathrm{co}(S_i)^e + \mathbb{Z}^2_+\right) \cap \mathbb{Z}^2$,

where $\mathrm{co}(S_i)^e$ is the set of extreme points of $\mathrm{co}(S_i)$. From (a) and Lemma 1, the set $\mathrm{co}(S_1)^e$ is finite. Denote $\mathrm{co}(S_1)^e$ by $\{x_0, x_1, \ldots, x_m\}$. We may assume

$$x_0^{(1)} > x_1^{(1)} > \cdots > x_m^{(1)}.$$

Thus, by the definition of S_1', we have

$$x_0^{(2)} < x_1^{(2)} < \cdots < x_m^{(2)}.$$

In addition, we have

$$0 > \frac{x_1^{(2)} - x_0^{(2)}}{x_1^{(1)} - x_0^{(1)}} > \frac{x_2^{(2)} - x_1^{(2)}}{x_2^{(1)} - x_1^{(1)}} > \cdots > \frac{x_m^{(2)} - x_{m-1}^{(2)}}{x_m^{(1)} - x_{m-1}^{(1)}}.$$

Here, $\frac{x_{i+1}^{(2)} - x_i^{(2)}}{x_{i+1}^{(1)} - x_i^{(1)}}$ means the slope of segment $x_i x_{i+1}$ (see figure 5). Therefore,

$$S_1 = \mathrm{co}\left(\mathrm{co}(S_1)^e + \mathbb{Z}^2_+\right) \cap \mathbb{Z}^2 = \bigcup_{i=0}^{m-1} \mathrm{co}(\{x_i, x_{i+1}\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2.$$

The set $\bigcup_{i=0}^{m-1} \mathrm{co}(\{x_i, x_{i+1}\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$ is composed of integral vectors that lie in the right upper area of the kinked line $x_0 x_1 \cdots x_m$.

---

[12] This lemma holds in any finite dimension.

Denote $\mathrm{co}(S_2)^e$ by $\{y_0, y_1, \ldots, y_n\}$. We may assume

$$y_0^{(1)} < y_1^{(1)} < \cdots < y_n^{(1)}.$$

Note that $\{y_0, y_1, \ldots, y_n\}$ is listed in ascending order with respect to the first coordinate, whereas $\{x_0, x_1, \ldots, x_m\}$ is listed in descending order. In a similar fashion, we have

$$\begin{aligned}
&y_0^{(2)} > y_1^{(2)} > \cdots > y_n^{(2)},\\
&\frac{y_0^{(2)} - y_1^{(2)}}{y_0^{(1)} - y_1^{(1)}} < \frac{y_1^{(2)} - y_2^{(2)}}{y_1^{(1)} - y_2^{(1)}} < \cdots < \frac{y_{n-1}^{(2)} - y_n^{(2)}}{y_{n-1}^{(1)} - y_n^{(1)}} < 0, \quad \text{and}\\
&S_2 = \mathrm{co}\left(\mathrm{co}(S_2)^e + \mathbb{Z}^2_+\right) \cap \mathbb{Z}^2 = \bigcup_{i=0}^{n-1} \mathrm{co}(\{y_i, y_{i+1}\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2.
\end{aligned}$$

Note that $\mathrm{co}(S_1 - \{x_0\} + S_2 - \{y_0\}) \cap \mathbb{Z}^2 = S_1 - \{x_0\} + S_2 - \{y_0\}$ if and only if $\mathrm{co}(S_1 + S_2) \cap \mathbb{Z}^2 = S_1 + S_2$. Thus, without loss of generality, we can assume $x_0 = y_0 = 0 \in \mathbb{Z}^2$. When $m = 0$ or $n = 0$, $S_1 + S_2$ is clearly discretely convex. Thus, in the rest of proof, we assume $m \geq 1$ and $n \geq 1$. Let $T_1 = \mathrm{co}(\{0, y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$. Then, T_1 is the set composed of integral vectors that lie in the right upper area of segment $0y_1$.

Claim 1 $\mathrm{co}(S_1 + T_1) \cap \mathbb{Z}^2 = S_1 + T_1$.

Proof of Claim 1. First, we consider the case where $x_1^{(2)}/x_1^{(1)} \leq y_1^{(2)}/y_1^{(1)}$. In this case, we have

$$\mathrm{co}(\{x_m, \ldots, x_0, y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2 = \left[\bigcup_{i=0}^{m-1} \mathrm{co}(\{x_i, x_{i+1}\} + \mathbb{Z}^2_+) \cup \mathrm{co}(\{0, y_1\} + \mathbb{Z}^2_+)\right] \cap \mathbb{Z}^2 = S_1 \cup T_1.$$

Therefore, $S_1 \cup T_1$ is discretely convex. As $0 \in S_1 \cap T_1$, we have $S_1 \cup T_1 \subset S_1 + T_1$. In addition, by the law of parallelogram, we have $S_1 + T_1 \subset S_1 \cup T_1$. Therefore, $S_1 \cup T_1 = S_1 + T_1$. Hence,

$$\mathrm{co}(S_1 + T_1) \cap \mathbb{Z}^2 = \mathrm{co}(S_1 \cup T_1) \cap \mathbb{Z}^2 = S_1 \cup T_1 = S_1 + T_1.$$

Next, we consider the case where $x_1^{(2)}/x_1^{(1)} > y_1^{(2)}/y_1^{(1)}$ (figure 5 depicts this case). Let

$$k = \max\left\{ j \in \{1, \ldots, m\} \;\middle|\; \frac{x_j^{(2)} - x_{j-1}^{(2)}}{x_j^{(1)} - x_{j-1}^{(1)}} > \frac{y_1^{(2)}}{y_1^{(1)}} \right\}.^{13}$$

Then,

$$\begin{aligned}
\frac{x_m^{(2)} - x_{m-1}^{(2)}}{x_m^{(1)} - x_{m-1}^{(1)}} &< \cdots < \frac{x_{k+1}^{(2)} - x_k^{(2)}}{x_{k+1}^{(1)} - x_k^{(1)}} \leq \frac{x_k^{(2)} - (x_k^{(2)} + y_1^{(2)})}{x_k^{(1)} - (x_k^{(1)} + y_1^{(1)})}\\
&< \frac{(x_k^{(2)} + y_1^{(2)}) - (x_{k-1}^{(2)} + y_1^{(2)})}{(x_k^{(1)} + y_1^{(1)}) - (x_{k-1}^{(1)} + y_1^{(1)})} < \cdots < \frac{(x_1^{(2)} + y_1^{(2)}) - y_1^{(2)}}{(x_1^{(1)} + y_1^{(1)}) - y_1^{(1)}}.
\end{aligned}$$

---

[13] In figure 5, $k = 3$.

Hence,

$$\begin{aligned}&\mathrm{co}(\{x_m, \dots, x_k, x_k + y_1, \dots, x_0 + y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2\\ &= \left[\bigcup_{i=k}^{m-1} \mathrm{co}(\{x_i, x_{i+1}\} + \mathbb{Z}^2_+) \cup \mathrm{co}(\{x_k, x_k + y_1\} + \mathbb{Z}^2_+) \cup \bigcup_{j=0}^{k-1} \mathrm{co}(\{x_j + y_1, x_{j+1} + y_1\} + \mathbb{Z}^2_+)\right] \cap \mathbb{Z}^2.\end{aligned}$$

Denote by E the set in the right-hand side of the above equation. That is, E is composed of elements of $\mathbb{Z}^2$ that lie in the right upper area of the kinked line $x_m \cdots x_k(x_k+y_1)\cdots(x_0+y_1)$. It is clear that $S_1 + T_1 \subset E$. From $0 \in S_1 \cap T_1$, it follows that

$$\left[\bigcup_{i=0}^{m-1} \mathrm{co}(\{x_i, x_{i+1}\} + \mathbb{Z}^2_+) \cup \mathrm{co}(\{0, y_1\} + \mathbb{Z}^2_+)\right] \cap \mathbb{Z}^2 = S_1 \cup T_1 \subset S_1 + T_1.$$

Therefore, in order to obtain the inclusion $E \subset S_1 + T_1$, it suffices to show that every integral vector in parallelograms $x_j(x_j+y_1)(x_{j-1}+y_1)x_{j-1}$, $1 \le j \le k$, belongs to S_1+T_1. Let $z \in \mathbb{Z}^2$ be in the parallelogram $x_j(x_j+y_1)(x_{j-1}+y_1)x_{j-1}$ with $1 \le j \le k$. As $\{x_j + y_1\} + \mathbb{Z}^2_+ \subset S_1 + T_1 + \mathbb{Z}^2_+ = S_1 + T_1$, we may assume $z \notin \{x_j + y_1\} + \mathbb{Z}^2_+$. When $z^{(1)} < x_j^{(1)} + y_1^{(1)}$, we have $z - x_j \in T_1$. Thus, $z = x_j + (z - x_j) \in S_1 + T_1$. On the other hand, when $z^{(2)} < x_j^{(2)} + y_1^{(2)}$, we have $z - (x_{j-1} + y_1) + x_{j-1} \in S_1$. Thus, $z = z - (x_{j-1} + y_1) + x_{j-1} + y_1 \in S_1 + T_1$. Hence, we have $E \subset S_1 + T_1$. Therefore, $E = S_1 + T_1$. As E is discretely convex, we have

$$\mathrm{co}(S_1 + T_1) \cap \mathbb{Z}^2 = \mathrm{co}(E) \cap \mathbb{Z}^2 = E = S_1 + T_1.$$

This completes the proof of Claim 1. ■

Let $T_2 = \mathrm{co}(\{y_1, y_2\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$. Note that $T_1 \cup T_2 = \mathrm{co}(\{0, y_1, y_2\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$.

Claim 2 $T_1 \cup T_2 = T_1 + \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$.

Proof of Claim 2. As $0 \in \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$, we have $T_1 \subset T_1 + \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$. Let $z \in T_2$. Then, $z - y_1 \in \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$, and therefore, $z = y_1 + (z - y_1) \in T_1 + \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$. Thus, we have $T_2 \subset T_1 + \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$. Hence, we have obtained $T_1 \cup T_2 \subset T_1 + \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$.

Next, we show the converse inclusion. Let $v \in T_1$ and $w \in \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2$. Then, $v+w \in \mathrm{co}(\{0, y_1, y_2, y_2-y_1\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2$. As $y_2-y_1 \in \mathrm{co}(\{0, y_1, y_2\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2$, we have $v+w \in \mathrm{co}(\{0, y_1, y_2\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2 = T_1\cup T_2$. Thus, $T_1+\mathrm{co}(\{0, y_2-y_1\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2 \subset T_1\cup T_2$. This completes the proof of Claim 2. ■

From Claim 2, it follows that

$$S_1 + (T_1 \cup T_2) = (S_1 + T_1) + \mathrm{co}(\{0, y_2 - y_1\} + \mathbb{Z}^2_+) \cap \mathbb{Z}^2.$$

As S_1+T_1 is discretely convex and $S_1+T_1+\mathbb{Z}^2_+ = S_1+T_1$ holds, from the same argument as Claim 1, the set $(S_1+T_1)+\text{co}(\{0, y_2-y_1\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2$ is discretely convex. Thus,

$$\text{co}\,(S_1+(T_1\cup T_2))\cap\mathbb{Z}^2 = S_1+(T_1\cup T_2).$$

Let $T_j = \text{co}(\{y_{j-1}, y_j\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2$ for every $j\in\{1,\dots,n\}$. Then, $S_2=\bigcup_{j=1}^n T_j$. By repeating same steps as the above, we have

$$\text{co}(S_1+S_2)\cap\mathbb{Z}^2 = S_1+S_2.$$

Thus, we have obtained the required result when S_1 and S_2 are both bounded from below.

Finally, we consider the general case. Let $z\in\text{co}(S_1+S_2)\cap\mathbb{Z}^2$. From Carathéodory's theorem, there are $\{x_1,x_2,x_3\}\subset S_1$ and $\{y_1,y_2,y_3\}\subset S_2$ such that $z\in\text{co}(\{x_1+y_1, x_2+y_2, x_3+y_3\})$. Thus,

$$\begin{aligned} z &\in \text{co}(\{x_1+y_1,x_2+y_2,x_3+y_3\})\cap\mathbb{Z}^2 \\ &\subset \text{co}(\{x_1,x_2,x_3\}+\{y_1,y_2,y_3\})\cap\mathbb{Z}^2 \\ &\subset \text{co}\left[\text{co}(\{x_1,x_2,x_3\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2+\text{co}(\{y_1,y_2,y_3\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2\right]\cap\mathbb{Z}^2. \end{aligned}$$

Note that $X=\text{co}(\{x_1,x_2,x_3\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2$ and $Y=\text{co}(\{y_1,y_2,y_3\}+\mathbb{Z}^2_+)\cap\mathbb{Z}^2$ are both discretely convex and bounded from below. In addition, $X+\mathbb{Z}^2_+=X$ and $Y+\mathbb{Z}^2_+=Y$. Therefore, from the above argument,

$$z\in\text{co}(X+Y)\cap\mathbb{Z}^2 = X+Y\subset S_1+S_2.$$

Thus, $\text{co}(S_1+S_2)\cap\mathbb{Z}^2\subset S_1+S_2$. This completes the proof of Theorem 3. ■

Appendix C

In this appendix, we show some facts stated in Example 3.

Proposition 1 *For every $i\in\{2,3\}$, the preference relation of agent i cannot be represented by a utility function u_i of the form*

$$u_i(x)=\sum_{j=1}^{3} w_i^j(x^{(j)}),$$

where every satisfaction function $w_i^j:\mathbb{R}_+\to\mathbb{R}$ is concave.

Proof. We now show this about agent 2. In a similar manner, we can show this about agent 3. Note that

$$u_2(2,0,0)>u_2(0,1,1)>u_2(1,1,0)>u_2(1,0,1).$$

Suppose that agent 2's preference relation can be represented as the sum of concave satisfaction functions. Then, there exist concave functions w^1, w^2, and w^3 on $\mathbb{R}_+$ such that

$$w^1(2)+w^2(0)+w^3(0) > w^1(0)+w^2(1)+w^3(1) > w^1(1)+w^2(1)+w^3(0) > w^1(1)+w^2(0)+w^3(1).$$

Therefore, we have

$$w^1(2) + w^3(0) > w^1(1) + w^3(1) \tag{1}$$

and

$$w^1(0) + w^3(1) > w^1(1) + w^3(0). \tag{2}$$

As w^1 is concave, we have

$$2w^1(1) - w^1(0) \geq w^1(2).$$

By substituting this into inequality (1), we have

$$w^1(1) + w^3(0) > w^1(0) + w^3(1).$$

This contradicts inequality (2). Therefore, agent 2's preference relation cannot be represented as the sum of concave satisfaction functions. ■

Proposition 2

$$\begin{aligned} U_1 &= \{(1,0,0),(0,1,0)\} + \mathbb{Z}^3_+, \\ U_2 &= \{(2,0,0),(0,1,1)\} + \mathbb{Z}^3_+, \ \textit{and} \\ U_3 &= \{(1,0,1),(0,2,0)\} + \mathbb{Z}^3_+. \end{aligned}$$

Proof. We show that $U_3 = \{(1,0,1),(0,2,0)\} + \mathbb{Z}^3_+$. In a similar manner, we can show the other equalities. Note that, for $x \in \mathbb{Z}^3_+$, $x \in \{(1,0,1),(0,2,0)\} + \mathbb{Z}^3_+$ if and only if $(2,1,0) \cdot x \geq 2$ and $(0,1,2) \cdot x \geq 2$. Therefore, if $x \in \{(1,0,1),(0,2,0)\} + \mathbb{Z}^3_+$, then

$$\min\{1, (2,1,0)\cdot x - 1, (0,1,2)\cdot x - 1\} = 1 \text{ and } \min\{1, (2,1,0)\cdot x - 1, (0,1,1)\cdot x\} = 1.$$

Hence, $u_3(x) \geq 0+1+1-1 = 1$. Thus, we have shown that $\{(1,0,1),(0,2,0)\}+\mathbb{Z}^3_+ \subset U_3$.

We now show the converse inclusion. Suppose that, for some $x \in U_3$, $\min\{1, (2,1,0)\cdot x - 1, (0,1,2)\cdot x - 1\} < 1$. Then, either $(2,1,0)\cdot x \leq 1$ or $(0,1,2)\cdot x \leq 1$. When $(2,1,0)\cdot x \leq 1$, we have $\min\{1, (2,1,0)\cdot x - 1, (0,1,1)\cdot x\} \leq 0$, and therefore, $u_3(x) \leq 1+0+0-1 = 0$. This contradicts $x \in U_3$. When $(0,1,2)\cdot x \leq 1$, we have $x^{(2)} \leq 1$ and $x^{(3)} = 0$, and therefore, $\min\{1, (1,0,0)\cdot x, (0,0,1)\cdot x\} = 0$. Thus, $u_3(x) \leq 0+0+1-1 = 0$. This contradicts $x \in U_3$. Hence, if $x \in U_3$, then $\min\{1, (2,1,0)\cdot x - 1, (0,1,2)\cdot x - 1\} =$

1. From the equivalence stated at the beginning of the proof, this means that $U_3 \subset \{(1,0,1),(0,2,0)\} + \mathbb{Z}^3_+$. ■

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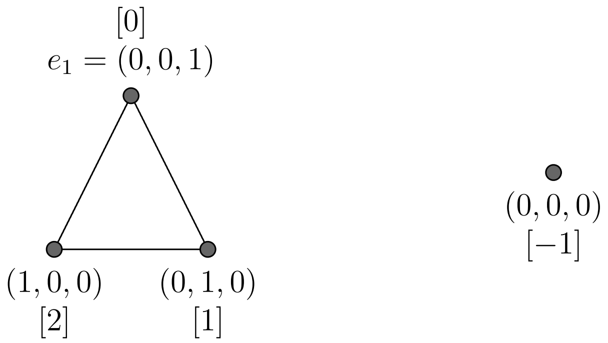

Figure 1: Agent 1's utility level

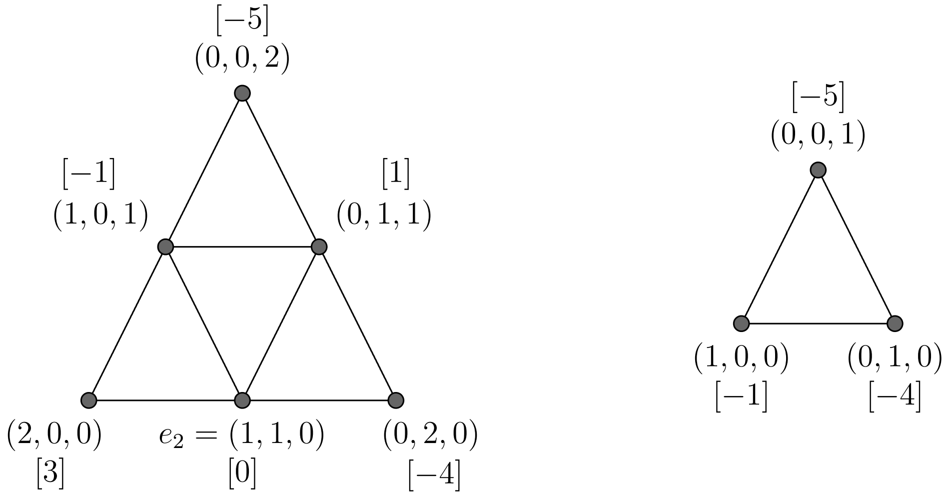

Figure 2: Agent 2's utility level

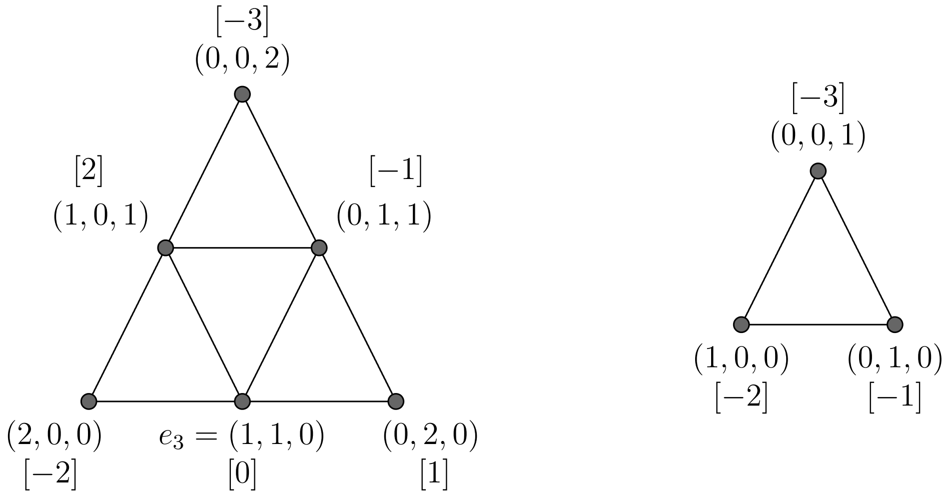

Figure 3: Agent 3's utility level

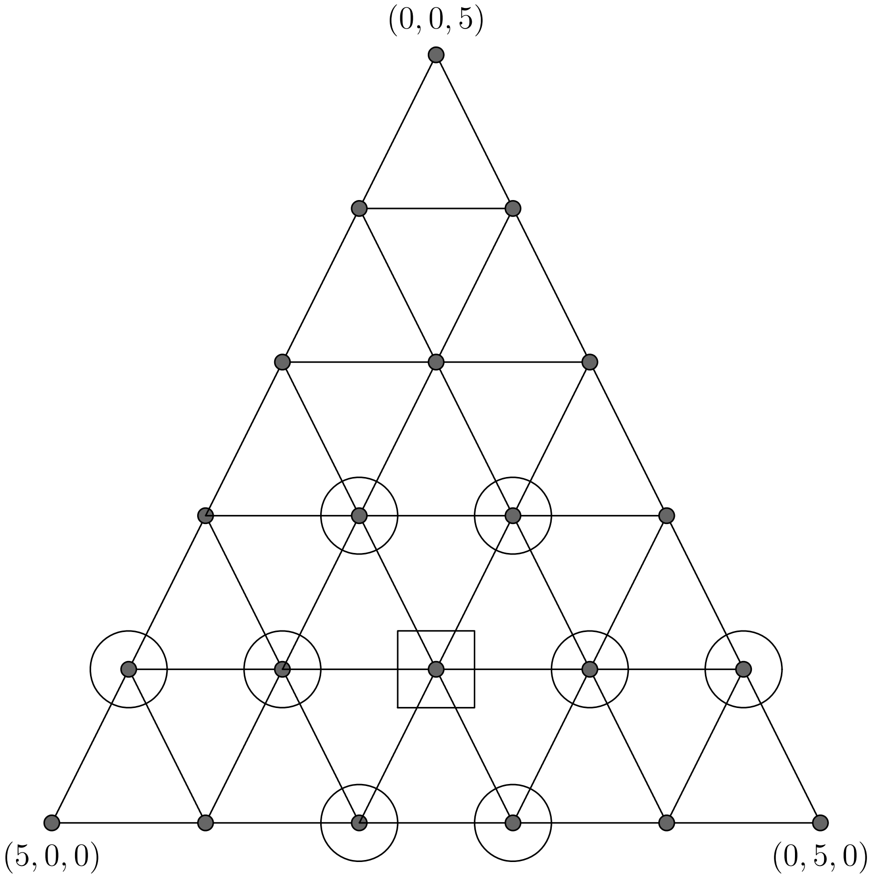

Figure 4: Minimal elements of $U_1 + U_2 + U_3$ and point $e_1 + e_2 + e_3$

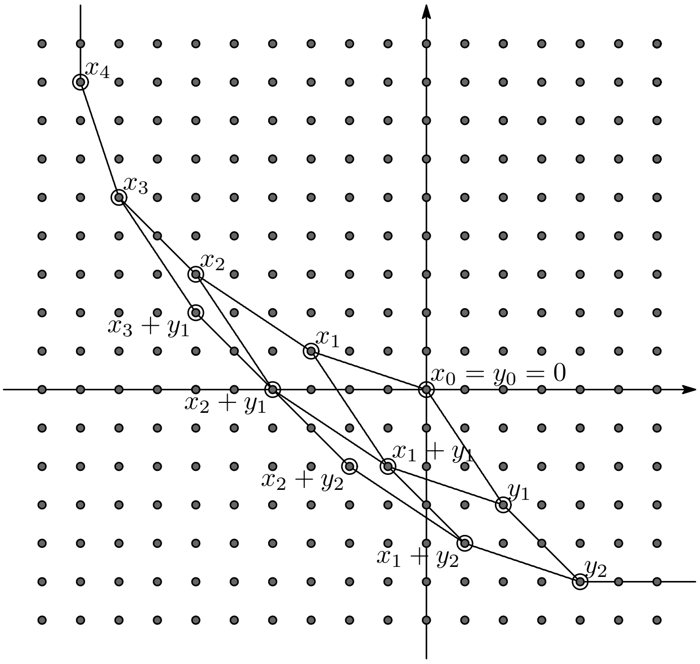

Figure 5: $S_1 + S_2$