Equilibrium social hierarchies:
a non-cooperative ordinal status game

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by

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Abstract

The article analyzes a non-cooperative game in strategic form, where each player's payoff depends on his action and his social status, which is given by his rank-order in the actions distribution. It examines the relationship between the degree of heterogeneity among status-seeking players and the distribution of their Nash equilibrium actions. The approach stands out because it does not use the heuristic assumption of a continuum of players. The latter is usual in the literature on status, although it is hard to reconcile with the widely accepted view that people's sensitivity to status is particularly relevant in small local environments. Our finding of different types of Nash equilibria brings forward that the role of social status can integrate both economic and sociological explanations of human behaviour. Basically, if differences among players are large, their equilibrium actions diverge; if differences are small, their equilibrium actions are the same. The article also shades the well-known claim that status seeking is socially inefficient by examining the Pareto efficiency of Nash equilibria. Finally, its key results are illustrated with a brief discussion of the impact of status seeking on savings behaviour.

Keywords: social status, Nash equilibria, game in strategic form, discontinuous payoff functions, Pareto efficiency, saving

JEL-code: C72, D11, D61, D62, E21, Z10

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1 Introduction

Over the last decades, a new literature on modelling consumer behaviour has emerged that stresses the links between individual choices. One series of studies explores the notion that, in many areas, individuals care for their relative position in the consumption or income hierarchy. It brings forward a diversity of results that challenge orthododox economic thinking, such as the relativity of economic growth for happiness (Hirsch, 1976; Easterlin, 2001; Layard, 2005), the tendency to overconsume and to neglect saving (Duesenberry, 1949; Frank, 1985b; Corneo and Jeanne, 1998), and the formation of class structure (Akerlof, 1997; Oxoby, 2003, 2004).[1] Although an innate desire for relative position, or status, is seen by some economists as inconsistent with rational self-interest, most agree that having high relative standing is often instrumental in achieving absolute goals.[2] Thus relative position may be an important argument in the reduced-form utility function.

The article addresses an elementary question that, to some surprise, has not been analysed thoroughly. It is concerned with the relation between the heterogeneity of status-seeking consumers and the distribution of their actions. For example, if consumers differ only with respect to income: what is the relationship between the income distribution and the distribution of status consumption goods? The problem is essentially game-theoretic. When every person's utility depends on how his action compares with the actions of other people, the choice of action becomes strategic, because everyone must anticipate the behaviour of the others in making his optimal decision. Here the literature usually assumes that each person chooses his optimal action, given the actions of all the others. For answering our question, it is therefore natural to resort to Nash equilibrium analysis.

The role of heterogeneity in situations where relative position matters requires to be more fully elucidated. Many studies that deal with the status seeking phenomon assume that all individuals are identical (e.g., Congleton, 1989; Akerlof, 1997; Corneo and Jeanne, 1998). They produce the result that, although *ex ante* individuals wish to perform better than others, *ex post* their actions turn out to be the same. This outcome is not very satisfactory, since it is difficult to see why individuals would continue to strive for a higher place in the hierarchy if they never observed such a hierarchy in practice. Studies that do assume individual differences, and so (often implicitly) address the relation between consumer heterogeneity and status consumption pattern, follow a partial-equilibrium approach (e.g., Layard, 1980; Frank, 1985b; Clark and Oswald, 1998), and thus run the risk that the generated distribution of actions differs from the hypothesized distribution upon which individuals base their actions. Moreover, they tend to ignore the possibility of multiple equilibrium distributions. It is true that the presumption that a unique equilibrium exists is warranted for those studies that employ a measure of status that accounts for the actions of other people by taking simply their arithmetic mean. In principle,

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[1] A classic is Veblen (1899), though the interest in relative position has an even longer history in economics (see e.g. Mason, 1998). More formal analyses appear with Morgenstern (1948), Duesenberry (1949), and Leibenstein (1950). For empirical evidence, see e.g. Solnick and Hemenway (1998), Ball et al. (2001), and Luttmer (2005). Probably the most comprehensive study is Robert Frank's *Choosing the Right Pond* (1985a).

[2] For example, higher relative standing may give access to better jobs and education. Frank (1985a) gives evidence of both innate desires for status and status-dependent opportunities (see also Hirsch, 1976, and Konrad, 1990). Postlewaite (1998) provides a methodological discussion on this point.

we then have an aggregative game, which under general conditions guarantees the existence of a unique Nash equilibrium (see e.g. Corchón, 2001). Yet we will argue that the empirical relevance of this measure of status is debatable. It implies, for example, that the gain in status for a person who improves his action is independent of the number of people he outstrips.

The approach taken here is most close to the seminal article of Frank (1985b) and the more recent contributions of Hopkins and Kornienko (2004a, 2004b). We follow these authors by adopting essentially the same ordinal measure of social status: a person's status is determined by the fraction of people who take an equal or lower social action than him.[3] However, we take a separate road in two respects. We give up the heuristic assumption of a continuous distribution of preference (or income) characteristics and thus an uncountably infinite number of agents. Although utterly unrealistic – there are not as many people in the world as there are points on a line, a continuum of agents – the continuity assumption is often mathematically convenient and very helpful in studying concepts like the "atomless" competitive market, where individual agents have no "mass". Our point is, however, that status seekers resemble much more the suppliers in an oligopolistic market, who do have mass. As cogently argued by Frank (1985a), people's sensitivity to social rank is particularly relevant in small local environments (e.g., among friends, relatives, colleagues, club members, or neighbours), so the size of a person's reference group is essentially limited.[4]

Our second point of departure relates to the nature of the studied equilibria. Since we assume that status is given by the fraction of people who take an equal or lower action, status is measured by way of a cumulative distribution function. The cited works of Frank and Hopkins and Kornienko, which employ such functions, ignore the fact that the resulting equilibrium distribution of actions is generally not unique.[5] One of our aims is to investigate the existence and uniqueness of equilibrium distributions of actions. We define a simple non-cooperative game in strategic form, where each player has a payoff function with two arguments: his action and his social status (as specified above). Differences among players may induce allocations with a distribution of actions and thus a distribution of social ranks, and the question is how this relationship exactly looks like. To that end, we study the Nash equilibria of the game; the associated distributions of social ranks can be considered as equilibrium social ladders or equilibrium social hierarchies.

Allocating the players to homogeneity classes, we identify two types of Nash equilibria: "fully diverse equilibria", where members of different classes take different actions, and "clustering equilibria", where the members of two or more classes take the same action. Whereas the literature only pays attention to fully diverse equilibria, it is the finding of clustering equilibria that is really special. It is at odds with the economic literature on status seeking, and even more so with the standard economic model. Where the latter typically implies that people with divergent preferences or opportunities act differently, we find that – if they care for status and are able to alter it – people may actually choose to do the same thing. Perhaps this is indeed the typical manifestation of status seeking: people imitating the status consumption pattern of those with higher incomes. It anyhow agrees with basic sociological notions that emphasize the uniformity of human behaviour. Note that our finding is not based on some kind of conformism, that is, on the assumption that people wish to conform to other people's actions (see, e.g., Jones,

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[3] Similar measures are employed by e.g. Layard (1980), Kosicky (1987), Robson (1992), Loch et al. (2000), and Corneo and Jeanne (2001).

[4] Neumark and Postlewaite (1998) find strong evidence that, because of relative income concerns, women's decisions to work are positively related to their sisters' employment decisions. Specifically, women whose sisters were employed in the previous year are about 10 to 15 percent more likely to work than those whose sisters did not work, for sisters living near by to one another.

[5] Hopkins and Kornienko (2004a, 2004b) seem to be aware of this, but restrict their attention to what they call symmetric equilibria (our "fully diverse equilibria" – see below).

1984, and Akerlof, 1997), but on the contrary assumes that people try to distance themselves from the actions of others. Moreover, we will show that uniform behaviour only arises if the underlying differences among people are small. In this sense, the existence of clustering equilibria captures basic elements from both economics and sociology.

A major reason why economists are interested in the phenomenon of status seeking arises from the claim that it is not socially efficient. The idea is that if the actions of people are motivated by social comparisons, the resulting equilibrium will be inherently sub-optimal because people ignore the externalities their actions create with respect to the relative standings of others.[6] As a potential correction for the externality, some authors have considered consumption taxes, income distribution policies, and institutional reforms (Layard, 1980, 2005; Frank, 1985a, 1997; Schor, 1998; Hopkins et al., 2004a). We take up this subject and examine the Pareto efficiency of the two mentioned types of Nash equilibria. Our conclusion is that the asserted sub-optimality of status seeking does not hold in a number of important cases. For example, if consumers are sufficiently heterogeneous, which is not unlikely if their reference group is relatively small, there exists a Nash equilibrium that is unique, fully diverse, and Pareto efficient. The principle argument is that by dropping the continuity assumption, the possibility arises that a consumer can alter his action without surpassing any other higher-ranked individual and thus without improving his social status, so that the costs of changing his action beyond the socially optimal level may be prohibitive. None the less we also find that the "sociological" clustering equilibria are generally not Pareto efficient.

Most of our key results will be illustrated with a brief discussion of the impact of status seeking on savings behaviour. Following in the footsteps of Duesenberry (1949), Frank (1985b) argues that consumers demand more status-enhancing or positional goods and fewer non-positional goods, as compared with a situation where they cannot alter their status. Because "savings" may be regarded as a non-positional good, status seeking thus reduces the consumer's average propensity to save. This effect is strongest for low-income consumers, so saving rates are not only lower across all income levels but also fall when moving to the lower tail of the income distribution. The latter agrees with the observed positive correlation between saving rates and income – something that is hard to reconcile with the Life-Cycle/Permanent-Income Hypothesis (cf. Dynan et al., 2004). We verify these two effects on saving rates, and find that Frank's assertions are particularly relevant for clustering equilibria. This configuration of equilibria indeed shows lower saving rates across the board and, within clusters of income classes, a positive relationship between saving rates and income.

The organization of the article is as follows. Section 2 starts with some background for our modelling of social status and then specifies the status game and introduces the basic assumptions, definitions, and terminology. It also presents an illustration of the game. Section 3 studies the status game with only two players. This limited setting already exhibits a number of features that characterize the general case of an arbitrary number of players. Section 4 contains the central results of the article. Because the discontinuities of the payoff functions are "bad", standard existence results from the literature cannot be applied, and therefore the analysis is carried out almost completely from scratch. Section 5 illustrates our main findings with a brief discussion of the impact of status seeking on savings behaviour. Section 6 concludes and suggests extensions.

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[6] The externalities are not always limited to the members of the reference group (the players of the game). Congleton (1989) elaborates the point that status seeking activities may harm but also benefit individuals who are not involved (think of e.g. the players and the spectators of a football match).

2 An ordinal status game

As introduction, consider this simple setting with status seeking. Person i chooses a single observable action x^i of some economic or social kind to maximize (reduced-form) utility:

$$u^i(x^i, s^i) \tag{1}$$

given a status production function:

$$s^i = s^i(x^i; \mathbf{x}^{\hat{\imath}}) \tag{2}$$

where s^i refers to his social status and $\mathbf{x}^{\hat{\imath}}$ is the vector of actions of all other people. Utility is influenced directly as well as indirectly. Loosely speaking, the action yields the normal *intrinsic* benefits and, via the status production function, it also generates *status* by establishing a certain position in the social hierarchy. People may differ in their subjective valuation of intrinsic gains and social status.[7] As usual in this literature, each person chooses his utility-maximizing action, given the actions of all the others.[8]

The status production functions need, of course, further specification. Just to fix ideas, let us suppose that status is earned by higher rather than lower actions. To capture the phenomenon of status seeking, at least in an *ex ante* sense, it is then natural to require that $s^i(x^i; \mathbf{x}^{\hat{\imath}})$ is increasing in x^i and decreasing in each component of $\mathbf{x}^{\hat{\imath}}$. Thus, if person i increases his action or if one or more others decrease their actions, he may move upward in the social hierarchy and acquire more status. One additional assumption is often implicitly made for convenience, although it is not always unrealistic: $s^i(x^i; \mathbf{x}^{\hat{\imath}}) = s^j(x^j; \mathbf{x}^{\hat{\jmath}})$ if $x^i = x^j$. So, two people i and j have the same status if they take the same action. It implies that the impact of other people is independent of who does what and who is affected, so that everyone is of equal social importance.

At this point, the formal literature takes two different paths with respect to the measurement of status. A "cardinal" measure of status is employed by studies that relate individual actions to some standard, often an average of other people's actions.[9] It agrees with the idea that status depends on a linear combination of the social differences $x^i - x^j$, or on a multiplicative combination of the social ratio's $\frac{x^i}{x^j}$. For example, the status of person i is continuous and strictly increasing in $\sum_j (x^i - x^j)$ and thus also in $x^i - \bar{x}^{\hat{\imath}}$, where $\bar{x}^{\hat{\imath}}$ is the arithmetic mean of the actions of the others.[10]

Relating individual actions to some average action, as in the example, has three features. First, moving closer to the action of a higher-status person, without matching his action, still raises one's status. Second, whether or not to match or to surpass the action of a higher-status person is a marginal decision. And third, the spread of other people's actions around the average plays no role. Particularly the last two features are probably too special. In small local environments, where people tend to engage in face-to-face contacts and have a name, it is likely that a person who surpasses someone else experiences a jump rather than a marginal increase

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[7] The precise shape of a person's status production function is exogenous for him; in reality, what gains status and by how much are the collective product of people's social valuations of each other. Although someone may try to change his preference for status, perhaps by psycho therapy, he is not able to change the collectively determined status production function (unless he is seen as trendsetter).

[8] The setting considered here does not address the signaling approach to status seeking, which generally assumes that a person's status depends on his absolute rather than relative level of signaling investments (see e.g. Frank, 1985b; Ireland, 1994; Bagwell and Bernheim, 1996).

[9] Examples are Duesenberry (1949), Boskin and Sheshinsky (1978), Layard (1980), Congleton (1989), Clark and Oswald (1998), and Oxoby (2003, 2004).

[10] This approach leads to (Cournot-like) aggregative games, for which there are well-established results about existence and uniqueness of Nash equilibria (see e.g. Corchón, 2001). Also note that a similar example for the social ratio's $\frac{x^i}{x^j}$ uses the geometric mean (this clearly ignores the possibility that one or more agents take no action ($x^j = 0$)).

in his social status. Typically the status production function s^i is then discontinuous in x^i at each component of the vector $\mathbf{x}^{\hat{\imath}}$. The third characteristic produces the unlikely outcome that the gain in status for a person who raises his action is independent of the number of people he outstrips. For suppose that, in the preceding example, person i increases his action from $\bar{x}^{\hat{\imath}}$ to $\bar{x}^{\hat{\imath}} + 2$. If he goes from (say) 10 to 12, his gain in case one half of the others chooses 5 and the other half 15, is exactly the same as in case one half of the others chooses 9 and the other half 11. However, in the latter case he surpasses one half of the people and even leaves everybody behind.

These peculiarities are avoided when status is measured in an "ordinal" way. Status seeking is then like racing: the winner has only to be faster than the person coming in second; nothing is gained by increasing his lead. An ordinal measure is perhaps also closer to the sociological literature, where status generally refers to rank-ordered relationships among people, as exemplified by the metaphor "social ladder" (see, e.g., Ridgeway and Walker, 1995). Studies that follow this path relate individual actions to the cumulative distribution of other people's actions.[11] In particular, the status of person i depends negatively on the fraction of people with strictly greater actions:

$$\frac{\#\{j \neq i \mid x^j > x^i\}}{N-1} \tag{3}$$

where # means "the number of elements of" and N is the total number of people. It agrees with the notion that people tend to look upward, rather than downward, when making comparisons.[12]

Although cardinal differences in social actions may have an impact in many occasions,[13] we think that ordinal differences always play a role and, therefore, the measurement of status requires at least an ordinal dimension. Therefore, the article examines the maximization of (1) for the ordinal status production function:

$$s^i(x^i; \mathbf{x}^{\hat{\imath}}) = 1 - \frac{\#\{j \neq i \mid x^j > x^i\}}{N-1} = \frac{\#\{j \neq i \mid x^j \leq x^i\}}{N-1}. \tag{4}$$

Thus we assume that a person's status is simply equal to the fraction of people he leaves behind plus those who are on a par with him. Basic assumptions, definitions, and terminology of the game are introduced in Section 2.1. Here we also derive some preliminary results that will prove useful in the subsequent analysis. An illustration of the game is given in Section 2.2.

2.1 The game

For a given integer $N \geq 2$, let $\mathcal{N} := \{1, \ldots, N\}$ and $Q := \{q_1, \ldots, q_N\}$, where $q_i := \frac{i-1}{N-1}$ $(i \in \mathcal{N})$. The foregoing suggests the following non-cooperative game:

Definition 1 *An (ordinal) status game is a game in strategic form with $N \geq 2$ players where each player $i \in \mathcal{N}$ has*

- *action set $X^i := [0, L^i]$, where $L^i > 0$; and*

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[11] Examples are more scarce, but include Layard (1980), Frank (1985b), Kosicki (1987), Robson (1992), Loch et al. (2000), Corneo and Jeanne (2001), Hopkins and Kornienko (2004a, 2004b). They all assume an infinity of agents, whose characteristics are continuously distributed in a measure space. As argued in the Introduction, the continuity assumption is problematic for the study of status seeking.

[12] That people look upward seems well-established (see, e.g., Duesenberry, 1949, or Frank, 1985a). The same observation underlies the welfare-economic notion of envy (Varian, 1974).

[13] A cardinal measure may be incorporated by using the difference between action x^i and the truncated mean of all the higher actions.

- *a payoff function* $v^i : \mathbf{X} \to \mathbb{R}$ *given by*

$$v^i(\mathbf{x}) := u^i(x^i, \frac{\#\{j \in \mathcal{N}\backslash\{i\} \mid x^j \leq x^i\}}{N-1}), \tag{5}$$

  where $\mathbf{X} := X^1 \times \cdots \times X^N$, $\mathbf{x} = (x^1, \ldots, x^N)$, *and the function* $u^i : X^i \times Q \to \mathbb{R}$ *is continuous and strictly quasi-concave in the first variable and strictly increasing in the second variable. Moreover, the function* u^i *is such that*[14]

$$\max u^i(\cdot, 0) > u^i(L^i, 1). \quad \diamond \tag{6}$$

The assumptions of the game are quite general and, in particular, do not require that u^i is differentiable in the action variable (the first one). Also, the status variable (the second one) is discrete: given the actions of the others, a higher action of player i that matches or surpasses the action level of one or more other players raises his status level (with a jump). Condition (6) seems special, but is only made for the sake of relevance. Before showing this, it is convenient to look for strongly dominated strategies.

Let $\hat{x}^i(s)$ denote the unique maximizer of the function $u^i(\cdot, s)$ ($i \in \mathcal{N}$ and $s \in Q$).[15] Then, for each status level $s \in Q$, the function $u^i(\cdot, s)$ is strictly increasing on the segment $[0, \hat{x}^i(s)]$ and strictly decreasing on the segment $[\hat{x}^i(s), L^i]$. Now suppose player i takes action $x < \min \hat{x}^i$ and fix the actions of the other players. Then x is strongly dominated by $\min \hat{x}^i$, since raising his action to $\min \hat{x}^i$ would always yield a higher payoff, even when his status level would not increase. There is also an upper bound. We can define (since the set is compact and non-empty)

$$\overline{L}^i := \max\{x^i \in X^i \mid u^i(x^i, 1) \geq u^i(\hat{x}^i(0), 0)\}. \tag{7}$$

Note that $0 \leq \min \hat{x}^i \leq \hat{x}^i(0) \leq \max \hat{x}^i \leq \overline{L}^i \leq L^i$. Then any action $x > \overline{L}^i$ is strongly dominated by $\hat{x}^i(0)$, because, whatever the other players' actions $\mathbf{x}^{\hat{i}}$ are, we find $v^i(x; \mathbf{x}^{\hat{i}}) \leq u^i(x, 1) < u^i(\hat{x}^i(0), 0)$. Here $u^i(\hat{x}^i(0), 0)$ is clearly the payoff player i can always secure. In sum, actions $\min \hat{x}^i$ and $\overline{L}^i$ border his relevant action set:

Proposition 2 *Each action* x^i *of player* i *with* $x^i < \min \hat{x}^i$ *or* $x^i > \overline{L}^i$ *is strongly dominated.* $\diamond$

Now we can explain the significance of condition (6) with the following lemma (the lemmas are proved in Appendix A):

Lemma 3 *Condition (6) implies*
(i) $u^i(\overline{L}^i, 1) = u^i(\hat{x}^i(0), 0)$;
(ii) $\hat{x}^i(s) < \overline{L}^i$ $(s \in Q)$. $\diamond$

So the condition ensures that, for all $s \in Q$, the function $u^i(\cdot, s)$ is strictly decreasing on a non-degenerate segment $[\hat{x}^i(s), \overline{L}^i]$ that is, moreover, large enough to make the game economically interesting. The condition essentially says that, given the actions of the others, each player eventually faces a trade-off between attaining a higher social position and gathering more intrinsic benefits. For instance, holding the actions of the others constant, suppose player i increases his action from x to $x' > x$, improving his status level from s to $s' \geq s$. The change in payoff follows as $u^i(x', s') - u^i(x, s) = [u^i(x', s') - u^i(x', s)] + [u^i(x', s) - u^i(x, s)]$. The first term in brackets

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[14] See the next footnote.

[15] A maximizer exists and is unique, since $u^i(\cdot, s)$ is continuous and strictly quasi-concave on a non-empty compact subset of $\mathbb{R}$.

relates to the gain in status and the second term to the loss in intrinsic benefits, provided that $x \geq \hat{x}^i(s)$. Of course, the trade-off ceases to be relevant when $x' > \overline{L}^i$ (recall $\overline{L}^i \leq L^i$).

Besides the function $\hat{x}^i$, the analysis of the game employs two other basic functions. Given $a, b \in Q$ with $a \leq b$, there exists for each player $i \in \mathcal{N}$ a unique $\bar{x}^i(b,a) \in [\hat{x}^i(b), L^i)$ such that

$$u^i(\bar{x}^i(b,a), b) = u^i(\hat{x}^i(a), a)$$

(see Figure 1). To see that $\bar{x}^i$ is well-defined, note that $u^i(\hat{x}^i(b), b) \geq u^i(\hat{x}^i(a), a)$ and, by virtue of (6), $u^i(L^i, b) \leq u^i(L^i, 1) < u^i(\hat{x}^i(0), 0) \leq u^i(\hat{x}^i(a), a)$ and that $u^i(\cdot, b)$ is continuous and strictly decreasing on $[\hat{x}^i(b), L^i]$. The following properties of $\bar{x}^i$ can also be verified with the help of Figure 1:

Lemma 4 *For each player $i \in \mathcal{N}$, it holds*
(i) $a = b \Rightarrow \bar{x}^i(b,a) = \hat{x}^i(b) = \hat{x}^i(a)$ $(b \in Q)$;
(ii) $a < b \Rightarrow \bar{x}^i(b,a) > \max\{\hat{x}^i(a), \hat{x}^i(b)\}$ $(a, b \in Q)$;
(iii) $\bar{x}^i$ is strictly increasing in its first variable[16] *and strictly decreasing in its second variable.* ◇

The third basic function is more general and contains $\bar{x}^i$ as a special case. It is particularly useful when we study games with three or more players. Let

$$V_s^i := [\hat{x}^i(s), \bar{x}^i(s,0)] \quad (s \in Q) \tag{8}$$

and note that, because $\bar{x}^i(s,0) \leq \bar{x}^i(1,0) = \overline{L}^i$, it holds $\cup_{s \in Q} V_s^i \subseteq [\min \hat{x}^i, \overline{L}^i]$. We call V_s^i the *V-set* of player i. Given $a, b \in Q$ with $a \leq b$ and $y \in V_a^i$, there exists for each player $i \in \mathcal{N}$ a unique $\bar{x}_+^i(b,a,y) \in V_b^i$ such that

$$u^i(\bar{x}_+^i(b,a,y), b) = u^i(y, a)$$

(see Figure 2).[17] Note that $\bar{x}_+^i(b,a,y) = \bar{x}^i(b,a)$ if $y = \hat{x}^i(a)$. Some other important properties are stated below.

Lemma 5 *For each player $i \in \mathcal{N}$, it holds*
(i) $a = b \Rightarrow \bar{x}_+^i(b,a,y) = y$ $(b \in Q, y \in V_b^i)$;
(ii) $a < b \Rightarrow \bar{x}_+^i(b,a,y) > \max\{y, \hat{x}^i(b)\}$ $(a, b \in Q, y \in V_a^i)$;
(iii) $c \leq a \leq b \Rightarrow \bar{x}_+^i(b,a,\bar{x}^i(a,c)) = \bar{x}^i(b,c)$ $(a, b, c \in Q)$;
(iv) $\bar{x}_+^i$ is strictly increasing in its first variable, strictly decreasing in its second variable,[18] *and strictly increasing in its third variable.* ◇

Lemma 6 *For any two players $i, j \in \mathcal{N}$, it holds*
(i) $\bar{x}_+^i = \bar{x}_+^j \Rightarrow \bar{x}^i = \bar{x}^j$;
(ii) $\bar{x}^i = \bar{x}^j \Rightarrow \hat{x}^i = \hat{x}^j$. ◇

The three functions $\hat{x}^i$, $\bar{x}^i$, and $\bar{x}_+^i$ are the main auxiliary objects of our exposition. Action $\hat{x}^i(s)$ is the optimal action of player i under status level s. If $s' > s$ is some higher status level that can be attained by the higher action $\bar{x}^i(s', s) > \hat{x}^i(s)$, then player i is indifferent between $\hat{x}^i(s)$ and $\bar{x}^i(s', s)$. And, more generally, if $y \in V_s^i$ is a certain action of player i that yields status s and if $s' > s$ is some higher status level that can be attained by the higher action $\bar{x}_+^i(s', s, y) > y$, then player i is indifferent between y and $\bar{x}_+^i(s', s, y)$. Here is a numerical example.

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[16] More precisely, for all $a, b_1, b_2 \in Q$ with $a \leq b_1 < b_2$, $\bar{x}^i(b_1, a) < \bar{x}^i(b_2, a)$.

[17] To see that $\bar{x}_+^i$ is well-defined, note that $u^i(\hat{x}^i(b), b) \geq u^i(\hat{x}^i(a), a) \geq u^i(y, a)$ and, from (6), $u^i(\bar{x}^i(b,0), b) = u^i(\hat{x}^i(0), 0) = u^i(\bar{x}^i(a,0), a) \leq u^i(y, a)$ and $u^i(\cdot, b)$ is continuous and strictly decreasing on V_b^i.

[18] More precisely, for all $a_1, a_2, b \in Q$ with $a_1 < a_2 \leq b$ and $y \in V_{a_1}^i \cap V_{a_2}^i$, $\bar{x}_+^i(b, a_1, y) > \bar{x}_+^i(b, a_2, y)$.

Example 7 *Suppose* $u^i(x,s) := - \mid x - A^i \mid + B^i s + C^i$, *where* $A^i, B^i, C^i > 0$ *and* $A^i + B^i \leq L^i$. *Note that* u^i *is indeed strictly quasi-concave in* x *and that, since* $\max u^i(\cdot, 0) = C^i$, *condition (6) is satisfied. Of course,*

$$\hat{x}^i = A^i.$$

Given $a, b \in Q$ *with* $a \leq b$, *we can solve*

$$- \mid \bar{x}^i - A^i \mid + B^i b + C^i = B^i a + C^i$$

for $\bar{x}^i \in [A^i, L^i)$. *Hence,*

$$\bar{x}^i(b,a) = A^i + B^i(b-a).$$

Also, given $a, b \in Q$ *with* $a \leq b$ *and* $y \in V_a^i = [A^i, A^i + B^i a]$, *we can solve*

$$- \mid \bar{x}_+^i - A^i \mid + B^i b + C^i = - \mid y - A^i \mid + B^i a + C^i$$

for $\bar{x}_+^i \in V_b^i = [A^i, A^i + B^i b]$. *Hence,*

$$\bar{x}_+^i(b,a,y) = y + B^i(b-a)$$

(the reader may verify Lemmas 4-6). $\diamond$

Consider now the status variable. With $\mathbf{z} = (z^1, \ldots, z^{N-1}) \in \mathbb{R}^{N-1}$, we define the function $F_{\mathbf{z}} : \mathbb{R} \to \mathbb{R}$ by

$$F_{\mathbf{z}}(x) := \frac{\#\{j \mid z^j \leq x\}}{N-1}. \tag{9}$$

Note that $F_{\mathbf{z}}$ is invariant for permutations of the coefficients of $\mathbf{z}$. Also, $F_{\mathbf{z}}$ is an increasing upper semi-continuous function with image in Q and the set of points where the function is discontinuous is $\{z^1, \ldots, z^{N-1}\}$. The discontinuities are even such that $F_{\mathbf{z}}$ is a step function. With $\mathbf{x} = (x^i; \mathbf{x}^{\hat{\imath}}) \in \mathbf{X}$,[19] the status production function of player i is defined as the function $s^i : \mathbf{X} \to \mathbb{R}$ given by

$$s^i(\mathbf{x}) := \frac{\#\{j \in \mathcal{N} \backslash \{i\} \mid x^j \leq x^i\}}{N-1} = F_{\mathbf{x}^{\hat{\imath}}}(x^i). \tag{10}$$

We call $s^i(\mathbf{x})$ and $F_{\mathbf{x}^{\hat{\imath}}}(x^i)$ the status of i in $\mathbf{x}$ (both notations will be used). Since $F_{\mathbf{x}^{\hat{\imath}}}$ is invariant for permutations of $\mathbf{x}^{\hat{\imath}}$, it does not matter which other player takes a certain action: all players are of equal social importance. The next lemma addresses the difference in status between two players.

Lemma 8 *Consider two players* $i, j \in \mathcal{N}$. *For each* $\mathbf{x} \in \mathbf{X}$,

$$F_{\mathbf{x}^{\hat{\imath}}}(x) - F_{\mathbf{x}^{\hat{\jmath}}}(x)$$

equals

$$\begin{array}{cll} 0 & \textit{if} & x^i = x^j \\ 0 & \textit{if} & x < x^i < x^j \\ 0 & \textit{if} & x^i < x^j \leq x \\ -1/(N-1) & \textit{if} & x^i \leq x < x^j. \quad \diamond \end{array}$$

The first result of this lemma has an immediate consequence, for it implies that if two players take the same action, they have the same status in $\mathbf{x}$. Even a stronger result holds:

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[19] Recall our vector notation: $\mathbf{x}^{\hat{\imath}} = (x^1, ..., x^{i-1}, x^{i+1}, ..., x^N)$, so $\mathbf{x}^{\hat{\imath}} \in \mathbb{R}^{N-1}$.

Proposition 9 *Given a multi-action* $\mathbf{x} \in \mathbf{X}$ *and two players* $i, j \in \mathcal{N}$*, it holds that* $s^i(\mathbf{x}) = s^j(\mathbf{x})$ *if and only if* $x^i = x^j$*.* $\diamond$

Proof. '$\Rightarrow$': Lemma 8 implies $x^i = x^j \Rightarrow s^i(\mathbf{x}) = s^j(\mathbf{x})$.

'$\Leftarrow$': By contradiction. We may suppose $x^j > x^i$. Since $F_{\mathbf{x}^{\hat{\jmath}}}$ is an increasing function, $F_{\mathbf{x}^{\hat{\jmath}}}(x^j) \geq F_{\mathbf{x}^{\hat{\jmath}}}(x^i)$ and, from Lemma 8, $F_{\mathbf{x}^{\hat{\jmath}}}(x^i) = F_{\mathbf{x}^{\hat{\imath}}}(x^i) + 1/(N-1) > F_{\mathbf{x}^{\hat{\imath}}}(x^i)$. Hence, $x^j > x^i \Rightarrow F_{\mathbf{x}^{\hat{\jmath}}}(x^j) > F_{\mathbf{x}^{\hat{\imath}}}(x^i) \Leftrightarrow s^j(\mathbf{x}) > s^i(\mathbf{x})$, which gives a contradiction. $\blacksquare$

Next, consider the players themselves. Using (10), the payoff function of player i becomes

$$v^i(\mathbf{x}) = u^i(x^i, s^i(\mathbf{x})). \tag{11}$$

Given the nature of the status variable, the payoff function v^i is discontinuous in x^i when the fixed actions of all other players are not all zero. It directly follows that also each payoff function is discontinuous. In fact, the discontinuities are so "bad" that the standard theorems on the existence of Nash equilibria cannot be applied. Yet each payoff function is upper semi-continuous, which, among other things, enables us to show that each status game has a Pareto-efficient multi-action (see Section 4.2). More precisely, we have

Proposition 10 *For given* $\mathbf{x}^{\hat{\imath}} \in \mathbf{X}^{\hat{\imath}}$*,*[20] *let* $g^i_{\mathbf{x}^{\hat{\imath}}} : X^i \to \mathbb{R}$ *be the conditional payoff function of player* i *defined by* $g^i_{\mathbf{x}^{\hat{\imath}}}(x^i) := u^i(x^i, s^i(x^i; \mathbf{x}^{\hat{\imath}})) = u^i(x^i, F_{\mathbf{x}^{\hat{\imath}}}(x^i))$*. Then*

- *the set of points where* $g^i_{\mathbf{x}^{\hat{\imath}}}$ *is discontinuous is* $(X^i \cap \{x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^N\}) \backslash \{0\}$*;*
- *each payoff function* v^i *is discontinuous and upper semi-continuous.* $\diamond$

Proof. First statement. Write $A^i := \{x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^N\}$. First we proof that $g^i_{\mathbf{x}^{\hat{\imath}}}$ is continuous in each point of X^i that does not belong to $(X^i \cap A^i) \backslash \{0\}$. Let a be such a point, so $a \in (X^i \backslash A^i) \cup \{0\}$. The function $g^i_{\mathbf{x}^{\hat{\imath}}}$ is a composition of the function $X^i \to \mathbb{R}^2$ given by $x \mapsto (x, F_{\mathbf{x}^{\hat{\imath}}}(x))$ and the function $X^i \times Q \to \mathbb{R}$ given by u^i. The set of points where the first function is continuous is $(X^i \backslash A^i) \cup \{0\}$. (Note that $F_{\mathbf{x}^{\hat{\imath}}} : [0, L^i] \to \mathbb{R}$ is continuous in 0 because it is upper semi-continuous in 0.) It follows that $g^i_{\mathbf{x}^{\hat{\imath}}}$ is continuous in each point of $(X^i \backslash A^i) \cup \{0\}$.

Now let a be a point in $(X^i \cap A^i) \backslash \{0\}$. We know that $F_{\mathbf{x}^{\hat{\imath}}}$ is discontinuous in a. Let w be the jump in a, i.e., $w := \lim_{x \downarrow a} F_{\mathbf{x}^{\hat{\imath}}}(x) - \lim_{x \uparrow a} F_{\mathbf{x}^{\hat{\imath}}}(x)$ (w is well-defined since $F_{\mathbf{x}^{\hat{\imath}}}$ is increasing). Because $F_{\mathbf{x}^{\hat{\imath}}}$ is discontinuous in a we have $w > 0$, and because $F_{\mathbf{x}^{\hat{\imath}}}$ is upper semi-continuous in a we have $\lim_{x \downarrow a} F_{\mathbf{x}^{\hat{\imath}}}(x) = F_{\mathbf{x}^{\hat{\imath}}}(a)$. Hence, $\lim_{x \uparrow a} F_{\mathbf{x}^{\hat{\imath}}}(x) = F_{\mathbf{x}^{\hat{\imath}}}(a) - w$. Note that $F_{\mathbf{x}^{\hat{\imath}}}(a) - w \in Q$. Let (a_n) be a sequence in X^i with $a_n < a$ for all n and with limit a. Then $\lim_{n \to \infty} F_{\mathbf{x}^{\hat{\imath}}}(a_n) = F_{\mathbf{x}^{\hat{\imath}}}(a) - w$. Because u^i is strictly increasing in its second variable and continuous, it follows that $g^i_{\mathbf{x}^{\hat{\imath}}}(a) = u^i(a, F_{\mathbf{x}^{\hat{\imath}}}(a)) > u^i(a, F_{\mathbf{x}^{\hat{\imath}}}(a) - w) = u^i(a, \lim_{n \to \infty} F_{\mathbf{x}^{\hat{\imath}}}(a_n)) = \lim_{n \to \infty} u^i(a_n, F_{\mathbf{x}^{\hat{\imath}}}(a_n)) = \lim_{n \to \infty} g^i_{\mathbf{x}^{\hat{\imath}}}(a_n)$. So $g^i_{\mathbf{x}^{\hat{\imath}}}(a) > \lim_{n \to \infty} g^i_{\mathbf{x}^{\hat{\imath}}}(a_n)$ and, therefore, $g^i_{\mathbf{x}^{\hat{\imath}}}$ is not continuous in a.

Second statement. Take $\mathbf{a} \in \mathbf{X}$. We prove that v^i is upper semi-continuous in $\mathbf{a}$. Fix $\epsilon > 0$. It is not difficult to see that s^i is upper semi-continuous. So there exists a neighbourhood $\mathbf{W} := W^1 \times \cdots \times W^N$ of $\mathbf{a}$ in $\mathbb{R}^N$ such that $s^i(\mathbf{x}) \leq s^i(\mathbf{a})$ ($\mathbf{x} \in \mathbf{X} \cap \mathbf{W}$). Since the function $u^i(\cdot, s^i(\mathbf{a}))$ is continuous, there exists a neighbourhood V^i of a^i in $\mathbb{R}$ such that

$$u^i(x^i, s^i(\mathbf{a})) \leq u^i(a^i, s^i(\mathbf{a})) + \epsilon \quad (x^i \in X^i \cap V^i).$$

Now $\mathbf{U} := (V^1 \cap W^1) \times \cdots \times (V^N \cap W^N)$ is a neighbourhood of $\mathbf{a}$ in $\mathbb{R}^N$, and for all $\mathbf{x} \in \mathbf{X} \cap \mathbf{U}$ we have

$$v^i(\mathbf{x}) = u^i(x^i, s^i(\mathbf{x})) \leq u^i(x^i, s^i(\mathbf{a})) \leq u^i(a^i, s^i(\mathbf{a})) + \epsilon = v^i(\mathbf{a}) + \epsilon.$$

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[20] We define $\mathbf{X}^{\hat{\imath}} := X^1 \times \cdots \times X^{i-1} \times X^{i+1} \times \cdots X^N$.

This completes our proof. ■

Players may have different payoff functions. These differences generally become consequential when they apply to the relevant action sets of players ($[\min \hat{x}^i, \overline{L}^i]$). How we distinguish between players on the basis of their payoff functions is a delicate issue, because, as will be seen, it influences our classification of equilibria into fully diverse and clustering equilibria. Although some arbitrariness cannot be avoided, we think the following definition is the most suitable.[21] Recalling that $\cup_{s\in Q} V^i_s \subseteq [\min \hat{x}^i, \overline{L}^i]$, we assume that any pair of players is either homogeneous or heterogeneous, according to

Definition 11 *Two players $i, j \in \mathcal{N}$ are homogeneous if*

$$V_s := V^i_s = V^j_s \quad (s \in Q)$$

and for each $s, s' \in Q$ and $x \in V_s$, $x' \in V_{s'}$

$$u^i(x, s) \geq u^i(x', s') \Leftrightarrow u^j(x, s) \geq u^j(x', s'). \tag{12}$$

Otherwise they are heterogeneous. ◇

Writing $i \sim j$ if players i and j are homogeneous, the relation $\sim$ is an equivalence relation on $\mathcal{N}$. Also note that players i and j are homogeneous if statement (12) holds over the same overall action set $X := X^i = X^j$.[22] Further, it can be shown that if i and j are homogeneous, a unique strictly increasing function f exists such that $u^j = f \circ u^i$ (see Proposition 42 in Appendix B). In an ordinal sense, therefore, homogeneous players have the same payoffs. Finally, it is important to observe that our definition of homogeneous players links up well with the key function $\bar{x}^i_+$, as indicated by

Proposition 12 *Given two players $i, j \in \mathcal{N}$, it holds that $\bar{x}^i_+ = \bar{x}^j_+$ if and only if i and j are homogeneous.* ◇

Proof. We prove that $i \sim j \Leftrightarrow \bar{x}^i_+ = \bar{x}^j_+$.

'⇒': Suppose $i \sim j$. Fix $a, b \in Q$ with $b \geq a$ and $y \in V^i_a$. We have

$$u^i(\bar{x}^i_+(b, a, y), b) = u^i(y, a).$$

Because $\bar{x}^i_+(b, a, y) \in V^i_b$ and $y \in V^i_a$, $i \sim j$ implies

$$u^j(\bar{x}^i_+(b, a, y), b) = u^j(y, a).$$

By definition, since $y \in V^j_a$, $\bar{x}^j_+(b, a, y)$ is the unique element $z \in V^j_b$ that solves

$$u^j(z, b) = u^j(y, a).$$

Since $\bar{x}^i_+(b, a, y) \in V^i_b = V^j_b$, it follows $\bar{x}^j_+(b, a, y) = \bar{x}^i_+(b, a, y)$.

'⇐': Suppose $\bar{x}^i_+ = \bar{x}^j_+$. Because the domains of these functions are the same, it holds $V_s := V^i_s = V^j_s$ $(s \in Q)$. Let $U := \cup_{s\in Q}\{u^i(x, s) \mid s \in Q, x \in V_s\}$. We will show that there exists a strictly increasing function $f : U \to \mathbb{R}$ such that $u^j(x, s) = (f \circ u^i)(x, s)$ $(s \in Q, x \in V_s)$. Then $i \sim j$ by Proposition 42 in Appendix B, and the proof is complete.

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[21] See the discussion in footnote 23 and Proposition 12 below.

[22] This is not obvious: see Proposition 43 in Appendix B.

Fix $z \in U$ and choose $s \in Q$ and $x \in V_s$ such that $u^i(x,s) = z$. Define

$$f(z) := u^j(x,s).$$

The function f does not depend on x and s, because if also $u^i(x',s') = z$, where $s' \in Q$ and $x' \in V_{s'}$, it holds $u^j(x,s) = u^j(x',s')$. To see this, suppose (without loss of generality) that $s' \geq s$. By definition, $u^i(x,s) = u^i(\bar{x}^i_+(s',s,x),s')$, so we have

$$u^i(\bar{x}^i_+(s',s,x),s') = u^i(x',s').$$

Because $\bar{x}^i_+(s',s,x), x' \geq \hat{x}^i(s')$, it follows

$$\bar{x}^i_+(s',s,x) = x'.$$

So, because $\bar{x}^i_+ = \bar{x}^j_+$,

$$\bar{x}^j_+(s',s,x) = x'.$$

Since $u^j(x,s) = u^j(\bar{x}^j_+(s',s,x),s')$ by definition, this gives $u^j(x,s) = u^j(x',s')$.

Therefore, by construction, we can write

$$u^j(x,s) = (f \circ u^i)(x,s) \quad (s \in Q, x \in V_s).$$

The function f is strictly increasing. For let $z, z' \in U$ with $z < z'$. Then, with $s, s' \in Q$, $x \in V_s$, and $x' \in V_{s'}$ such that $z = u^i(x,s)$ and $z' = u^i(x',s')$, it holds $u^i(x,s) < u^i(x',s')$. Because $i \sim j$, we have $u^j(x,s) < u^j(x',s')$, which implies $f(z) = u^j(x,s) < u^j(x',s') = f(z')$. ■

Differences among players are likely to induce allocations with a distribution of social ranks, and the question is how this relationship exactly looks like. To that end, we study the Nash equilibria of the game. Each player chooses his utility-maximizing action on the assumption that the actions of all the other players remain constant. In a Nash equilibrium, each player does so based on a correct forecast of the actions of all the others. Hence, it is a set of mutually consistent actions: a multi-action $\mathbf{x} \in \mathbf{X}$ such that for all $i \in \mathcal{N}$ and $z \in X^i$, $v^i(x^i;\mathbf{x}^{\hat{i}}) \geq v^i(z;\mathbf{x}^{\hat{i}})$. Denoting a Nash-equilibrium by $\mathbf{n} = (n^1,\ldots,n^N)$, Proposition 2 directly implies $n^i \in [\min \hat{x}^i, \overline{L}^i]$. We can even be more specific about the equilibrium action of player i:

Proposition 13 *Consider a status game. Then, for each Nash equilibrium* $\mathbf{n}$*, it holds* $n^i \in V^i_{s^i(\mathbf{n})}$ *(*$i \in \mathcal{N}$*).* ◇

Proof. Let $b := F_{\mathbf{n}^{\hat{i}}}(n^i)$, so $V^i_{s^i(\mathbf{n})} = V^i_{F_{\mathbf{n}^{\hat{i}}}(n^i)} = V^i_b = [\hat{x}^i(b), \bar{x}^i(b,0)]$. We prove $n^i \in V^i_b$ $(i \in \mathcal{N})$ by contradiction.

Suppose $n^i < \hat{x}^i(b)$. Then $F_{\mathbf{n}^{\hat{i}}}(\hat{x}^i(b)) \geq F_{\mathbf{n}^{\hat{i}}}(n^i) = b$. Using this, we have

$$v^i(n^i;\mathbf{n}^{\hat{i}}) = u^i(n^i,b) < u^i(\hat{x}^i(b),b) \leq u^i(\hat{x}^i(b), F_{\mathbf{n}^{\hat{i}}}(\hat{x}^i(b))) = v^i(\hat{x}^i(b);\mathbf{n}^{\hat{i}}),$$

a contradiction.

Suppose $n^i > \bar{x}^i(b,0)$. Then, from Lemma 4, also $n^i > \hat{x}^i(b)$. Using this, we have

$$v^i(n^i;\mathbf{n}^{\hat{i}}) = u^i(n^i,b) < u^i(\bar{x}^i(b,0),b) = u^i(\hat{x}^i(0),0) = v^i(\hat{x}^i(0);\mathbf{n}^{\hat{i}}),$$

a contradiction.

This completes our proof. ■

Moreover, the equilibrium action of player i just equals his optimal action under status level $s^i(\mathbf{n})$ or the equilibrium action of some other player(s):

Proposition 14 *Suppose* $\mathbf{n}$ *is a Nash equilibrium of a status game. Then for all* $i \in \mathcal{N}$*, it holds* $n^i \in \{\hat{x}^i(s^i(\mathbf{n}))\} \cup \{n^1, \dots, n^{i-1}, n^{i+1}, \dots, n^N\}$. $\diamond$

Proof. By contradiction. So, fix i and, with $Z := \{n^1, \dots, n^{i-1}, n^{i+1}, \dots, n^N\}$, suppose $n^i \notin \{\hat{x}^i(s^i(\mathbf{n}))\} \cup Z$. Consider the conditional payoff function $g^i_{\mathbf{n}^{\hat\imath}}$ defined in Proposition 10. Because $n^i \notin Z$, there exists $\delta > 0$ such that $F_{\mathbf{n}^{\hat\imath}}$ is constant, say s, on $(n^i - \delta, n^i + \delta)$. It follows that $g^i_{\mathbf{n}^{\hat\imath}}(x^i) = u^i(x^i, s)$ on $X^i \cap (n^i - \delta, n^i + \delta)$. Because $n^i \neq \hat{x}^i(s)$, there exists $\eta > 0$ such that $\hat{x}^i(s) \notin (n^i - \eta, n^i + \eta)$. Now, consider $g^i_{\mathbf{n}^{\hat\imath}}$ on $X^i \cap (n^i - \epsilon, n^i + \epsilon)$, where $\epsilon := \min\{\delta, \eta\}$. Because this restricted function is strictly quasi-concave and does not have the maximizer $\hat{x}^i(s)$ in its domain, it is strictly monotone and so does not have a maximizer. In particular, n^i is not a maximizer of the restricted function and, therefore, also not of the unrestricted function $g^i_{\mathbf{n}^{\hat\imath}}$. This is a contradiction. $\blacksquare$

One particular issue is whether, in a Nash equilibrium, homogeneous players always act the same and heterogeneous players always act differently. Remarkably, the answer to the latter question is no: sometimes heterogeneous players behave differently, sometimes they do the same. It requires a complicated analysis to identify the precise conditions determining the equilibrium behaviour of heterogeneous players. The answer to the former question is yes: homogeneous players indeed act the same. This result is already stated in the next theorem.

Theorem 15 *Consider a status game and suppose* i *and* j *are homogeneous players. Then, for each Nash equilibrium* $\mathbf{n}$*, it holds* $n^i = n^j$. $\diamond$

Proof. We proceed by contradiction: suppose $n^i \neq n^j$ (with $i \neq j$). We may also suppose $n^i < n^j$. Note first that, by Lemma 8 (first and fourth result),

$$F_{\mathbf{n}^{\hat\imath}}(n^j) = F_{\mathbf{n}^{\hat\jmath}}(n^j) \ \text{ and } F_{\mathbf{n}^{\hat\imath}}(n^i) = F_{\mathbf{n}^{\hat\jmath}}(n^i) - \frac{1}{N-1}.$$

Now, because $\mathbf{n}$ is a Nash-equilibrium, $v^i(n^i; \mathbf{n}^{\hat\imath}) \geq v^i(n^j; \mathbf{n}^{\hat\imath})$ and $v^j(n^j; \mathbf{n}^{\hat\jmath}) \geq v^j(n^i; \mathbf{n}^{\hat\jmath})$. Therefore,

$$u^i(n^i, F_{\mathbf{n}^{\hat\imath}}(n^i)) \geq u^i(n^j, F_{\mathbf{n}^{\hat\imath}}(n^j)) = u^i(n^j, F_{\mathbf{n}^{\hat\jmath}}(n^j)),$$

or, with $s := F_{\mathbf{n}^{\hat\imath}}(n^i)$ and $s' := F_{\mathbf{n}^{\hat\jmath}}(n^j)$,

$$u^i(n^i, s) \geq u^i(n^j, s').$$

Because i and j are homogeneous, and noting that $n^i \in V^i_s$ and $n^j \in V^j_{s'}$ by Proposition 13, this implies

$$u^j(n^i, s) \geq u^j(n^j, s').$$

But we also have

$$\begin{aligned} u^j(n^j, s') &= u^j(n^j, F_{\mathbf{n}^{\hat\jmath}}(n^j)) \geq u^j(n^i, F_{\mathbf{n}^{\hat\jmath}}(n^i)) = u^j(n^i, F_{\mathbf{n}^{\hat\imath}}(n^i) + \frac{1}{N-1}) \\ &> u^j(n^i, F_{\mathbf{n}^{\hat\imath}}(n^i)) = u^j(n^i, s) \end{aligned}$$

which is absurd. $\blacksquare$

Let us allocate the players to M equivalence classes, hereafter called *homogeneity classes.* So members of the same homogeneity class are always homogeneous and members from different classes heterogeneous. Noting that $1 \leq M \leq N$, the case $M = 1$ will be referred to as *complete homogeneity* and the case $M = N$ as *complete heterogeneity.*

Theorem 15 directly implies that the number of different actions in a Nash equilibrium is at most M. Hence, only equilibria with M or with less than M different action levels can exist. Equilibria with M different actions show a one-to-one correspondence between action level and class membership. Equilibria with less than M different actions are such that the members of two or more homogeneity classes choose the same action level.

We will use the following terminology. Let E denote the set of Nash equilibria. For $K \in \{1, \ldots, N\}$, $E^{(K)}$ denotes the set of Nash equilibria with exactly K different actions:

$$E^{(K)} := \{\mathbf{n} \in E \mid \#\{n^1, \ldots, n^N\} = K\}. \tag{13}$$

Since $K \leq M$, it holds

$$E = E^{(1)} \cup \cdots \cup E^{(M)}$$

(of course, the union is disjoint). An element of $E^{(K)}$ is called *K-level equilibrium*. Specifically, an element of $E^{(1)}$ is called *single-level equilibrium*, an element of $E^{(M)}$ *fully diverse equilibrium*, and an element of $E \backslash E^{(M)}$ *clustering equilibrium.*[23]

Any Nash equilibrium implies a social hierarchy or social ladder in the following sense. With $\mathbf{a} = (a^1, \ldots, a^N) \in \mathbb{R}^N$, let the function $S_{\mathbf{a}} : \mathbb{R} \to \mathbb{R}$ be defined by $S_{\mathbf{a}}(x) := \#\{j \mid a^j \leq x\}/N$. Then, given a Nash equilibrium $\mathbf{n}$, $S_{\mathbf{n}}(n^i)$ indicates the relative position of player i among N players on the social ladder. It is closely related to the social status perceived by player i himself: $S_{\mathbf{n}}(n^i) = s^i(\mathbf{n})(N-1)/N + 1/N$. Thus we can say that the function $S_{\mathbf{n}}$ represents the social ladder implied by the equilibrium $\mathbf{n}$. Note that different Nash equilibria may entail the same social ladder. Also, by Theorem 15, an equilibrium with a non-degenerate social hierarchy, so with $K \geq 2$, requires at least two homogeneity classes. If there is only one, so in case of complete homogeneity, only a single-level equilibrium can exist (which is also a fully diverse equilibrium). But the presence of two or more homogeneity classes is not sufficient for a non-degenerate social hierarchy, since it may entail a single-level equilibrium (which is also a clustering equilibrium).

To be sure, until now nothing guarantees the existence of a Nash equilibrium. We have already remarked that the discontinuities of our payoff functions preclude the application of the standard existence theorems in the literature. So we cannot rely on general results, and have to carry out the analysis largely from scratch. Therefore, it is instructive to begin with studying a game with only two players, which is done in Section 3. The general analysis with N players follows in Section 4. But first we motivate the game with an illustration.

2.2 Illustration

We illustrate the status game with a simple model of consumer behaviour. The model also underlies our discussion of the impact of status seeking on saving in Section 5.[24]

Suppose there are N consumers ($N \geq 2$). Each consumer $i \in \mathcal{N}$ wishes to choose a combination of positional (or status-generating) goods and non-positional goods (e.g., safety devices, insurance, redemption of a mortgage, future consumption (see Frank, 1985a)), with quantities x^i and y^i, to maximize utility:

$$U(x^i, y^i; s^i)$$

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[23] Our taxonomy of equilibria clearly depends on Definition 11 and Theorem 15. The proof of Theorem 15 implicitly reveals that the equivalence relation $\sim$ on $\mathcal{N}$ is stronger than what is required for any pair of players to have always the same Nash equilibrium strategy. Only a symmetric relation R on $\mathcal{N}$ is needed with the property that iRj implies, for each Nash equilibrium $\mathbf{n}$, $u^i(n^i, s^i(\mathbf{n})) \geq u^i(n^j, s^j(\mathbf{n})) \Rightarrow u^j(n^i, s^i(\mathbf{n})) \geq u^j(n^j, s^j(\mathbf{n}))$. This more general approach would alter our classification of Nash equilibria. Yet, we have two problems with this approach: relation R is restricted to Nash strategies and, because $iRj \not\Leftrightarrow \bar{x}^i_+ = \bar{x}^j_+$, it lacks a convenient property (see Proposition 12). Finally, it may be noted that defining equivalence is basically a semantic issue.

[24] A continuous version of the model is studied by Frank (1985b); see also Hopkins and Kornienko (2004a).

given a status production function:

$$s^i = F_{\mathbf{x}^{\hat{\imath}}}(x^i)$$

and subject to a budget constraint:

$$p_x x^i + p_y y^i \leq w^i.$$

Here income $w^i > 0$ for all $i \in \mathcal{N}$, prices $p_x, p_y > 0$, and $U : \mathbb{R}^2_+ \times Q \to \mathbb{R}$ is such that, for each $s^i \in Q$, $U(\cdot, \cdot; s^i)$ is continuous, strictly increasing, and strictly quasi-concave.[25] Moreover, U is strictly increasing in the third variable. Note that, just for simplicity, we assume that consumers may differ only with respect to income.

This setting is a game in strategic form where each player i has a two-dimensional action set:

$$Z^i := \{(x^i, y^i) \in \mathbb{R}^2_+ \mid p_x x^i + p_y y^i \leq w^i\}$$

and, writing $z^i = (x^i, y^i)$,[26] a payoff function $f^i : \mathbf{Z} \to \mathbb{R}$ given by

$$f^i(\mathbf{z}) := U(x^i, y^i; F_{\mathbf{x}^{\hat{\imath}}}(x^i)).$$

If we impose one more restriction on the shape of U: for all $i \in \mathcal{N}$

$$\max_{(x^i,y^i)\in Z^i} U(\cdot, \cdot; 0) > U(\frac{w^i}{p_x}, 0; 1), \tag{14}$$

the game boils down to an ordinal status game, as will be shown now.

Since, for each $x^i \in \mathbb{R}_+$ and $s^i \in Q$, $U(x^i, \cdot; s^i)$ is strictly increasing, the budget constraint will hold with strict equality at any maximizer, so we can substitute for y^i in U and write

$$U(x^i, \frac{w^i - p_x x^i}{p_y}; s^i).$$

With $X^i := [0, L^i] := [0, w^i/p_x]$, we define the function $u^i : X^i \times Q \to \mathbb{R}$ by

$$u^i(x^i, s^i) := U(x^i, \frac{w^i - p_x x^i}{p_y}; s^i),$$

and arrive at

$$v^i(\mathbf{x}) = u^i(x^i, F_{\mathbf{x}^{\hat{\imath}}}(x^i)).$$

The function u^i is continuous and strictly quasi-concave in its first variable and strictly increasing in its second variable. Restriction (14) ensures that u^i also satisfies the relevance condition (6). Hence, we have indeed constructed an ordinal status game.

Let us call the earlier two-goods game Γ and the associated status game Γ'. It is easy to show that in the former game, there exists for each $s \in Q$ and $w > 0$ a unique maximizing pair, which we denote by $(\hat{x}(s, w), \hat{y}(s, w))$. Its counterpart in game Γ' is $\hat{x}^i(s)$, according to

Proposition 16 *Given the two-goods game* Γ *and the associated status game* Γ'*, it holds* $\hat{x}^i(s) = \hat{x}(s, w^i)$ $(i \in \mathcal{N}, s \in Q)$. $\diamond$

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[25] Here $\mathbb{R}_+ := [0, \infty)$. "Strictly increasing" means that for all $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$, we have $U(a_2, b_2; s) \geq U(a_1, b_1; s)$ whenever $a_2 \geq a_1$ and $b_2 \geq b_1$ and the inequality is strict whenever $a_2 > a_1$ and $b_2 > b_1$. Actually, $U(., .; s)$ needs to be strictly quasi-concave only on all budget lines (thus allowing for e.g. a Cobb-Douglas specification).

[26] Similarly, $\mathbf{z} = ((x^1, y^1), \ldots, (x^N, y^N))$ and $\mathbf{Z} := Z^1 \times \cdots \times Z^N$.

Proof. Because $p_x\hat{x}(s,w^i)+p_y\hat{y}(s,w^i) = w^i$, $\hat{x}(s,w^i)$ is the unique maximizer of the function $[0,w^i/p_x] \to \mathbb{R}$ given by $x \mapsto U(x,(w^i - p_x x)/p_y; s)$, i.e. the function $u^i(\cdot,s)$. Since $\hat{x}^i(s)$ denotes the unique maximizer of $u^i(\cdot,s)$, therefore, $\hat{x}^i(s) = \hat{x}(s,w^i)$. $\blacksquare$

This result will be implicitly employed in Section 5, as will be the next proposition, which indicates that the existence and Pareto efficiency of the Nash equilibria of both games are closely related:

Proposition 17 *Consider the two-goods game* Γ *and the associated status game* Γ'.

- *If* $\mathbf{z} = ((x^1,y^1),\ldots,(x^N,y^N)) \in \mathbf{Z}$ *is a Nash equilibrium of* Γ*, then* $\mathbf{x}$ *is a Nash equilibrium of* Γ'*. If* $\mathbf{z}$ *is even a Pareto-efficient Nash equilibrium, then* $\mathbf{x}$ *is also a Pareto-efficient Nash equilibrium.*
- *If* $\mathbf{x} \in \mathbf{X}$ *is a Nash equilibrium of* Γ'*, then, with* $y^i := (w^i - p_x x^i)/p_y$ $(i \in \mathcal{N})$*,* $\mathbf{z} := ((x^1,y^1),\ldots,(x^N,y^N))$ *is a Nash equilibrium of* Γ*. If* $\mathbf{x}$ *is even a Pareto-efficient Nash equilibrium, then* $\mathbf{z}$ *is also a Pareto-efficient Nash equilibrium.* $\diamond$

Proof. First statement. Suppose $\mathbf{z}$ is a Nash equilibrium of Γ. For $i \in \mathcal{N}$, (x^i,y^i) is then a maximizer of the function

$$h^i(\cdot;\mathbf{z}^{\hat{\imath}}) = U(\cdot,\cdot;s^i(\cdot;\mathbf{x}^{\hat{\imath}})).$$

Define $H^i := \{(x,y) \in \mathbb{R}^2 \mid p_x x + p_y y = w^i\}$ $(i \in \mathcal{N})$. Then, by Lemma 44 in Appendix B, $(x^i,y^i) \in H^i$. It follows that x^i is a maximizer of the function $X^i \to \mathbb{R}$ given by $a^i \mapsto U(a^i,(w^i - p_x a^i)/p_y; s^i(a^i;\mathbf{x}^{\hat{\imath}}))$, i.e. of the function $v^i(\cdot;\mathbf{x}^{\hat{\imath}})$. Thus $\mathbf{x}$ is a Nash equilibrium of Γ'.

"Pareto efficiency" by contradiction. So suppose $\mathbf{z} = ((x^1,y^1),\ldots,(x^N,y^N)) \in \mathbf{Z}$ is a Pareto-efficient Nash equilibrium of Γ, but $\mathbf{x}$ is not a Pareto-efficient Nash equilibrium of Γ'. Define $e^i := (w^i - p_x x^i)/p_y$ $(i \in \mathcal{N})$ and note that $e^i \geq y^i$ (since $(x^i,y^i) \in Z^i$). Let $\mathbf{a} \in \mathbf{X}$ be a Pareto improvement of $\mathbf{x}$, so $v^i(\mathbf{a}) \geq v^i(\mathbf{x})$ $(i \in \mathcal{N})$ with at least one of these inequalities strict, say for $i = k$. Define $b^i := (w^i - p_x a^i)/p_y$ $(i \in \mathcal{N})$ and let $\mathbf{p} := ((a^1,b^1),\ldots,(a^N,b^N)) \in \mathbf{Z}$. Because $(a^i,b^i),(x^i,e^i) \in H^i$ and $e^i \geq y^i$, we obtain

$$h^i(\mathbf{p}) = U(a^i,b^i;s^i(\mathbf{a})) = u^i(a^i,s^i(\mathbf{a})) = v^i(\mathbf{a}) \geq$$

$$v^i(\mathbf{x}) = u^i(x^i,s^i(\mathbf{x})) = U(x^i,e^i;s^i(\mathbf{x})) \geq U(x^i,y^i;s^i(\mathbf{x})) = h^i(\mathbf{z}),$$

so $h^i(\mathbf{p}) \geq h^i(\mathbf{z})$ and for $i = k$ a strict inequality. Therefore, $\mathbf{z}$ is not Pareto efficient – a contradiction.

Second statement. We have to prove that, for $i \in \mathcal{N}$, (x^i,y^i) is a maximizer of the function $h^i(\cdot;\mathbf{z}^{\hat{\imath}})$. Because $\mathbf{x}$ is a Nash equilibrium of Γ', x^i is a maximizer of the function $v^i(\cdot;\mathbf{x}^{\hat{\imath}})$, i.e. of the function $a^i \mapsto U(a^i,(w^i - p_x a^i)/p_y; s^i(a^i;\mathbf{x}^{\hat{\imath}}))$. Suppose (x^i,y^i) would *not* be a maximizer of the function $h^i(\cdot;\mathbf{z}^{\hat{\imath}})$. Then there exists $(c^i,d^i) \in Z^i$ with $h^i((c^i,d^i);\mathbf{z}^{\hat{\imath}}) > h^i((x^i,y^i);\mathbf{z}^{\hat{\imath}})$. Now, for $e^i := (w^i - p_x c^i)/p_y$ $(i \in \mathcal{N})$, we have $(c^i,e^i) \in H^i$, $e^i \geq d^i$, and $h^i((c^i,e^i);\mathbf{z}^{\hat{\imath}}) \geq h^i((c^i,d^i);\mathbf{z}^{\hat{\imath}})$. So

$$h^i((c^i,e^i);\mathbf{z}^{\hat{\imath}}) > h^i((x^i,y^i);\mathbf{z}^{\hat{\imath}}).$$

Because $(x^i,y^i),(c^i,e^i) \in H^i$, this becomes $h^i((c^i,(w^i - p_x c^i)/p_y);\mathbf{z}^{\hat{\imath}}) > h^i((x^i,(w^i - p_x x^i)/p_y);\mathbf{z}^{\hat{\imath}})$, or

$$U(c^i,(w^i - p_x c^i)/p_y; s^i(c^i;\mathbf{x}^{\hat{\imath}})) > U(x^i,(w^i - p_x x^i)/p_y; s^i(x^i;\mathbf{x}^{\hat{\imath}})),$$

a contradiction.

"Pareto efficiency" by contradiction. So suppose $\mathbf{x} \in \mathbf{X}$ is a Pareto-efficient Nash equilibrium of Γ', but $\mathbf{z} := ((x^1,y^1),\ldots,(x^N,y^N))$ is not a Pareto-efficient Nash equilibrium of Γ. Let $\mathbf{p} :=$

$((a^1,b^1),\dots,(a^N,b^N)) \in \mathbf{Z}$ be a Pareto improvement of $\mathbf{z}$, so $h^i(\mathbf{p}) \geq h^i(\mathbf{z})$ $(i \in \mathcal{N})$ and at least one of these inequalities is strict, say for $i = k$. Define $c^i := (w^i - p_x a^i)/p_y$ $(i \in \mathcal{N})$ and let $\mathbf{q} = ((a^1,c^1),\dots,(a^N,c^N))$. Because $c^i \geq b^i$, it follows that $U(a^i,c^i;s^i(\mathbf{a})) \geq U(a^i,b^i;s^i(\mathbf{a}))$. Now, because $(a^i,c^i),(x^i,y^i) \in H^i$,

$$\begin{aligned} v^i(\mathbf{a}) &= u^i(a^i,s^i(\mathbf{a})) = U(a^i,c^i;s^i(\mathbf{a})) \geq U(a^i,b^i;s^i(\mathbf{a})) = h^i(\mathbf{p}) \\ &\geq h^i(\mathbf{z}) = U(x^i,y^i;s^i(\mathbf{x})) = u^i(x^i,s^i(\mathbf{x})) = v^i(\mathbf{x}), \end{aligned}$$

so $v^i(\mathbf{a}) \geq v^i(\mathbf{x})$ and for $i = k$ a strict inequality. Therefore, $\mathbf{x}$ is not Pareto efficient – a contradiction. ■

3 The status game with only two players

In this section, we examine an ordinal status game with only two players. Understanding this restricted game is worthwhile, because it already exhibits a number of features that characterize the general case of N players, such as the presence of fully diverse and clustering equilibria. We also verify whether a Nash equilibrium is indeed Pareto inefficient, as claimed by the literature on status seeking (see Introduction). After re-stating the game in Section 3.1, we study the case of homogeneous players in Section 3.2 and that of heterogeneous players in Section 3.3.

3.1 The game

Consider a status game with two players, so $\mathcal{N} = \{1,2\}$. Their actions are given by the multi-action $\mathbf{x} = (x^1,x^2)$. Writing $i,j \in \mathcal{N}$ with $i \neq j$, the payoff function of player i is

$$v^i(\mathbf{x}) := u^i(x^i, \frac{\#\{j \mid x^j \leq x^i\}}{2-1}) = \begin{cases} u^i(x^i,0) & \text{if} \quad 0 \leq x^i < x^j \\ u^i(x^i,1) & \text{if} \quad x^j \leq x^i \leq L^i, \end{cases}$$

which is presented by the solid curve in Figure 3 (for arbitrary x^j).

It is convenient to work with reaction correspondences, although this concept will not be used later on.[27] The reaction correspondence of player i is $R^i : X^j \multimap X^i$ defined by

$$R^i(x^j) := \arg\max_{x^i} v^i(\mathbf{x}).$$

It states player i's optimal action(s) in response to any specified action of player j. Using $\hat{x}^i(1) < \bar{x}^i(1,0)$ (from Lemma 4 (*ii*)) and with the help of Figure 3, we find

$$R^i(x^j) = \begin{cases} \{\hat{x}^i(1)\} & \text{if} \quad 0 \leq x^j \leq \hat{x}^i(1) \\ \{x^j\} & \text{if} \quad \hat{x}^i(1) < x^j < \bar{x}^i(1,0) \\ \{x^j, \hat{x}^i(0)\} & \text{if} \quad x^j = \bar{x}^i(1,0) \\ \{\hat{x}^i(0)\} & \text{if} \quad x^j > \bar{x}^i(1,0). \end{cases} \tag{15}$$

Since $R^i(x^j) = \{x^i \mid v^i(x^i;x^j) \geq v^i(z;x^j); z \in X^i\}$, a Nash equilibrium is a multi-action $\mathbf{x} \in \mathbf{X}$ such that $x^1 \in R^1(x^2)$ and $x^2 \in R^2(x^1)$. An equilibrium indeed always exists (a proof is in Appendix A[28]):

Theorem 18 *A status game with two players has a Nash equilibrium.* ◇

Hereafter we identify these equilibria (there may be more than one). Note that (15) already implies a Nash equilibrium $\mathbf{n}$ has $n^i \in \{n^j, \hat{x}^i(0), \hat{x}^i(1)\}$, which is in line with Proposition 14.

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[27] Expressing the reaction correspondences analytically becomes increasingly laborious as the number of players goes up. For example, with 3 players, a reaction correspondence already contains 16 or more case expressions.

[28] A proof based on the reaction correspondences is rather cumbersome. The concise proof in Appendix A relies on some general results derived in Section 4.

3.2 Complete homogeneity

If the two players are homogeneous, then $\hat{x}^1 = \hat{x}^2$ and $\bar{x}^1 = \bar{x}^2$ (Proposition 12 and Lemma 6). It follows that $R^1 = R^2$ if $L^1 = L^2$. Below superscript i is omitted.

Figure 4 presents the reaction correspondence (15) of each player. Although the diagram has $\hat{x}(0) < \hat{x}(1)$, it is also possible that $\hat{x}(0) \geq \hat{x}(1)$; the only restriction is $\max\{\hat{x}(0), \hat{x}(1)\} < \bar{x}(1, 0)$ (Lemma 4 (*ii*)). It is easily verified that the distinction between these two cases bears no impact on the following results. A similar remark holds for the possibility that $\hat{x}(0) = 0$ or $\hat{x}(1) = 0$.

A Nash equilibrium is implied by the "intersection" of the graphs of the reaction correspondences in Figure 4. It is seen that a continuum of equilibria exists and that, in each equilibrium, the two players take the same action, as implied by Theorem 15. Hence, all equilibria are single-level equilibria. The set of Nash equilibria is given by $E = E^{(1)} = \{(a, a) \mid a \in [\hat{x}(1), \bar{x}(1, 0)]\}$.

This set can be ranked by the Pareto criterion. Since, in each equilibrium, both players have the highest possible status ($s = 1$) and an action equal to or greater than $\hat{x}(1)$, equilibria with higher action levels than $\hat{x}(1)$ bring lower payoffs. Hence, those with lower action levels are Pareto superior, and the Pareto-dominating equilibrium is $\mathbf{n} = (\hat{x}(1), \hat{x}(1))$. The latter is also Pareto efficient, because both players receive their maximum payoffs $u^1(\hat{x}(1), 1)$ and $u^2(\hat{x}(1), 1)$.

3.3 Complete heterogeneity

Figure 5 shows the reaction correspondences (15) of the two players if they are heterogeneous. Diagrams A-C present three typical configurations that may arise with respect to the existence of Nash equilibria.[29]

Diagram A illustrates that, even though the players are different, there may exist a continuum of equilibria where they take the same action. This set of single-level equilibria, now also clustering equilibria, is given by $E = E^{(1)} = \{(a, a) \mid a \in \cap_{i=1}^2 [\hat{x}^i(1), \bar{x}^i(1, 0)]\}$. As before, these equilibria can be ranked by the Pareto criterion. Those with lower action levels are superior, and the Pareto-dominating equilibrium is $(\hat{x}^2(1), \hat{x}^2(1))$. The latter is also Pareto efficient, because player 2 receives his maximum payoff and thus can only loose in case of a different action, while player 1 receives a lower payoff $u^1(x^1, 0) \leq u^1(\hat{x}^1(0), 0) = u^1(\bar{x}^1(1, 0), 1) < u^1(\hat{x}^2(1), 1)$ if he takes a low-status action $x^1 < \hat{x}^2(1)$, and a lower or equal payoff $u^1(x^1, 1) \leq u^1(\hat{x}^2(1), 1)$ if he takes a high-status action $x^1 \geq \hat{x}^2(1)$.

Diagram B shows a unique fully diverse equilibrium, where the players take different actions and enjoy different amounts of status. It reflects the most simple equilibrium social hierarchy. Formally, $E = E^{(2)} = \{(\hat{x}^1(0), \hat{x}^2(1))\}$. The equilibrium is Pareto efficient, since player 2 gathers the maximum payoff, while player 1 gets a lower or equal payoff $u^1(x^1, 0) \leq u^1(\hat{x}^1(0), 0)$ if he takes a low-status action $x^1 < \hat{x}^2(1)$, and a lower payoff $u^1(x^1, 1) \leq u^1(\hat{x}^2(1), 1) < u^1(\bar{x}^1(1, 0), 1) = u^1(\hat{x}^1(0), 0)$ if he takes a high-status action $x^1 \geq \hat{x}^2(1)$.

Diagram C shows a borderline case with a fully diverse equilibrium and a clustering equilibrium. Formally, $E = E^{(1)} \cup E^{(2)}$ with $E^{(1)} = \{(\hat{x}^2(1), \hat{x}^2(1))\}$ and $E^{(2)} = \{(\hat{x}^1(0), \hat{x}^2(1))\}$. Both equilibria are Pareto efficient for almost identical reasons as before. Moreover, they yield the same payoff to each player: $u^1(\bar{x}^1(1, 0), 1) = u^1(\hat{x}^1(0), 0)$ to player 1 and $u^2(\hat{x}^2(1), 1)$ to player 2. This configuration illustrates the general point that we can have equilibria with different numbers of action levels at the same time (see also Example 29).

Comparing the three diagrams of Figure 5, it is seen that the critical condition determining the configuration of Nash equilibria is whether $\hat{x}^2(1) < \bar{x}^1(1, 0)$ (Diagram A), $\hat{x}^2(1) > \bar{x}^1(1, 0)$ (Diagram B), or $\hat{x}^2(1) = \bar{x}^1(1, 0)$ (Diagram C). Here $\hat{x}^2(1)$ is the optimal action of player 2 under status level 1, and $\bar{x}^1(1, 0)$ is the action of player 1 that, if it yielded status level 1, would

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[29] As before, whether $0 \leq \hat{x}^i(0) < \hat{x}^i(1)$ or $0 \leq \hat{x}^i(1) < \hat{x}^i(0)$ ($i \in \mathcal{N}$) does not basically alter our analysis.

make him equally well off as the lower optimal action $\hat{x}^1(0)$ under status level 0. Recalling that $V_s^i := [\hat{x}^i(s), \bar{x}^i(s,0)]$ is the V-set of player i, the critical condition thus sets demands on how V_1^1 compares with V_1^2. To put it roughly, little heterogeneity of the players (some overlap of V_1^1 and V_1^2) gives rise to a continuum of clustering equilibria, comparable with homogeneity, and much heterogeneity (no overlap of V_1^1 and V_1^2) gives rise to a unique fully diverse equilibrium. This broad statement remains valid in the general case of N players.

4 General analysis

Building on our insights from Section 2.1, we will now continue with the analysis of the status game for any finite number of players ($N \geq 2$). The existence of a fully diverse equilibrium or a clustering equilibrium is studied in Section 4.1. Our key results are Propositions 27 and 28, which derive necessary and sufficient conditions for the existence of any type of Nash equilibrium. The Pareto efficiency of fully diverse and clustering equilibria is studied in Section 4.2.

4.1 Existence of fully diverse and clustering equilibria

4.1.1 On the set of single-level equilibria $E^{(1)}$

Even in case of complete heterogeneity, i.e. where the number of homogeneity classes M equals N, there may exist equilibria where all players take the same action. This is implied by

Theorem 19 *Consider a status game. Then* $E^{(1)} = \{(a,\dots,a) \mid a \in \cap_{i=1}^N V_1^i\}$. $\diamond$

Proof. Recall $V_1^i = [\hat{x}^i(1), \bar{x}^i(1,0)]$.

'$\subseteq$': Let $\mathbf{n} = (a,\dots,a) \in E^{(1)}$ and apply Proposition 13 with $s^i(\mathbf{n}) = F_{\mathbf{n}^{\hat{\imath}}}(n^i) = 1$ for all i.

'$\supseteq$': For $x^i \in [a, L^i]$, we have $\hat{x}^i(1) \leq a \leq x^i$, so $v^i(x^i;\mathbf{a}^{\hat{\imath}}) = u^i(x^i,1) \leq u^i(a,1) = v^i(a;\mathbf{a}^{\hat{\imath}})$. And for $x^i \in [0,a)$, we have $v^i(x^i;\mathbf{a}^{\hat{\imath}}) = u^i(x^i,0) \leq u^i(\hat{x}^i(0),0) = u^i(\bar{x}^i(1,0),1) \leq u^i(a,1) = v^i(a;\mathbf{a}^{\hat{\imath}})$. $\blacksquare$

A single-level equilibrium does not necessarily exist, because the segments V_1^i may have an empty intersection. But if all players are homogeneous, this is not possible, since the numbers $\hat{x}^i(1)$ and $\bar{x}^i(1,0)$ are then independent of i (Proposition 12 and Lemma 6). Writing in this case V_{11} (consistent with our notation later on) and recalling Theorem 15, we have

Corollary 20 *Consider a status game with complete homogeneity* ($M = 1$). *Then*

$$E = E^{(1)} = \{(a,\dots,a) \mid a \in V_{11}\},$$

so $\#E = \infty$. $\diamond$

Hence, with solely homogeneous players, there exists an infinite number of Nash equilibria, which are all single-level equilibria.

4.1.2 Additional terminology

To study the existence of Nash equilibria with more than one action level, the relations between action levels and subsets of players must be specified. We use the following terminology. For $K \in \{1,\dots,N\}$, define $\mathcal{K} := \{1,\dots,K\}$. An *ordered partition* of dimension K of $\mathcal{N}$ is a finite sequence $(\mathcal{C}_1,\dots,\mathcal{C}_K)$ of non-empty disjoint subsets of $\mathcal{N}$ whose union is $\mathcal{N}$. Let $\mathcal{C} = (\mathcal{C}_1,\dots,\mathcal{C}_K)$ be such an ordered partition. We define the function $\lambda : \mathcal{N} \to \mathcal{K}$ by

$$\lambda(i) \text{ is the unique element of } \mathcal{K} \text{ for which } i \in \mathcal{C}_{\lambda(i)}, \tag{16}$$

and call this the *label function* of $\mathcal{C}$.

Now consider a multi-action $\mathbf{x} \in \mathbf{X}$ and let $\#\{x^1, \dots, x^N\} = K$. Define real numbers $c_1, \dots, c_K$ by

$$\{x^1, \dots, x^N\} = \{c_1, \dots, c_K\}_{\neq},$$
$$c_1 < \dots < c_K.$$

With these K numbers, or *level data*, we can define for $k \in \mathcal{K}$

$$\mathcal{C}_k := \{i \in \mathcal{N} \, | x^i = c_k\}. \tag{17}$$

It follows that $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_K)$ is an ordered partition of $\mathcal{N}$ and that, with λ as the label function of $\mathcal{C}$, it holds $x^i = c_{\lambda(i)}$ $(i \in \mathcal{N})$. We call $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_K)$ the *cluster partition* of $\mathbf{x}$ and its elements *clusters*. For example, $\mathbf{x} = (4, 2, 7, 4, 8)$ has cluster partition $\mathcal{C} = (\{2\}, \{1, 4\}, \{3\}, \{5\})$. Note that the cluster partition $(\mathcal{C}_1, \dots, \mathcal{C}_K)$ of a vector $\mathbf{x}$ also tells us that all $i \in \mathcal{C}_k$ are on rung k of the implied social ladder.

In line with all this, we write $E_{\mathcal{C}}$ for the set of Nash equilibrium vectors with cluster partition $\mathcal{C}$. If $\mathcal{P}_K$ denotes the set of *all* ordered partitions of dimension K of $\mathcal{N}$, we have

$$E^{(K)} = \cup_{\mathcal{C} \in \mathcal{P}_K} E_{\mathcal{C}},$$
$$E = \cup_{K=1}^{K=N} \cup_{\mathcal{C} \in \mathcal{P}_K} E_{\mathcal{C}}.$$

Our final definition is used to characterize the degree of heterogeneity among players. Let $i, j \in \mathcal{N}$ be two heterogeneous players. Let $Q_{++} := Q \backslash \{0\}$. We say that the non-degenerate V-sets of i and j are *ordered disjoint sets* if[30]

$$V_s^i < V_s^j \ (s \in Q_{++}) \text{ or } V_s^i > V_s^j \ (s \in Q_{++}).$$

The definition assumes that i and j are heterogeneous (in particular, $i \neq j$), because if they were homogeneous, none of both systems of inequalities could hold. Also note that the non-degenerate V-sets of i and j are ordered disjoint sets if and only if

$$\bar{x}^i(s, 0) < \hat{x}^j(s) \ (s \in Q_{++}) \text{ or } \bar{x}^j(s, 0) < \hat{x}^i(s) \ (s \in Q_{++}).$$

Recalling that there are M homogeneity classes, the next lemma follows immediately:

Lemma 21 *Suppose for all pairs of heterogeneous players the non-degenerate V-sets are ordered disjoint sets. Then there exists a unique ordered partition $(\mathcal{C}_1, \dots, \mathcal{C}_M)$ of $\mathcal{N}$ where each $\mathcal{C}_k$ is a homogeneity class, such that – writing V_{ks} for the sets V_s^i $(i \in \mathcal{C}_k)$ (which are all equal) –*

$$V_{1s} < \dots < V_{Ms} \ (s \in Q_{++}). \quad \diamond$$

4.1.3 On the set of N-level equilibria $E^{(N)}$

We proceed with an analysis of the set $E^{(N)}$, so the set of Nash equilibria where all players take different actions. Theorem 15 already implies that, for such an equilibrium to exist, all players must be heterogeneous:

Corollary 22 *Consider a status game. If $E^{(N)} \neq \emptyset$, then there is complete heterogeneity (i.e. $M = N$).* $\diamond$

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[30] Here, for two subsets A and B of $\mathbb{R}$, we write $A < B$ if $a < b$ $(a \in A, b \in B)$. Note that $V_0^i = [\hat{x}^i(0), \bar{x}^i(0, 0)] = \{\hat{x}^i(0)\}$ by Lemma 4(i), so V_0^i is a degenerate set (a set with only one element).

Hence, each $\mathbf{n} \in E^{(N)}$ is a fully diverse equilibrium. Because $E^{(N)} = \cup_{\mathcal{C} \in \mathcal{P}_N} E_{\mathcal{C}}$ and the union is disjoint,[31] for each $\mathbf{n} \in E^{(N)}$ there exists a unique N-dimensional ordered partition of $\mathcal{N}$ such that $\mathbf{n} \in E_{\mathcal{C}}$. Note that the set $\mathcal{P}_N$ of such partitions contains $N!$ elements.

Now suppose $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_N)$ is such a partition. Then its label function $\lambda : \mathcal{N} \to \mathcal{N}$ is a permutation of $\mathcal{N}$, so we have $\mathcal{C}_i = \{\lambda^{-1}(i)\}$ $(i \in \mathcal{N})$ and, for $\mathbf{n} \in E^{(N)}$, we have for the level data: $c_1 = n^{\lambda^{-1}(1)} < \cdots < c_N = n^{\lambda^{-1}(N)}$, and for the status level of each player: $s^i(\mathbf{n}) = q_{\lambda(i)}$ $(i \in \mathcal{N})$. Applying Proposition 14, we thus find that if $E_{\mathcal{C}} \neq \emptyset$, then $E_{\mathcal{C}} = \{\hat{x}^1(q_{\lambda(1)}), \ldots, \hat{x}^N(q_{\lambda(N)})\}$. The next result also provides necessary and sufficient conditions for the existence of this equilibrium:[32]

Proposition 23 *Fix $\mathcal{C} \in \mathcal{P}_N$ and let $\lambda : \mathcal{N} \to \mathcal{N}$ be its label function. Then $E_{\mathcal{C}} \neq \emptyset$ if and only if the inequalities*

$$\bar{x}^{\lambda^{-1}(k)}(q_l, q_k) \leq \hat{x}^{\lambda^{-1}(l)}(q_l) \quad (k \in \mathcal{N}, l = k, \ldots, N)$$

hold. Moreover, if $E_{\mathcal{C}} \neq \emptyset$, then

$$E_{\mathcal{C}} = \{\hat{x}^1(q_{\lambda(1)}), \ldots, \hat{x}^N(q_{\lambda(N)})\}. \quad \diamond$$

In this fully diverse equilibrium, each player i takes his optimal action $\hat{x}^i(q_{\lambda(i)})$ under the status level $q_{\lambda(i)}$. The inequality conditions basically ensure that player i cannot increase his payoff by choosing one of the higher action levels with more status, taken by some other player.

Suppose the fully diverse equilibrium indeed exists, so $E_{\mathcal{C}} \neq \emptyset$. Are there other N-dimensional ordered partitions of $\mathcal{N}$ that can also give rise to fully diverse equilibria? Or a bit less formal: if we have an equilibrium social ladder with one person on each of the N rungs, can these people trade places and form another equilibrium social ladder with as many rungs? The answer is negative: the set $E^{(N)} = \cup_{\mathcal{C} \in \mathcal{P}_N} E_{\mathcal{C}}$ contains at most one element.

Theorem 24 *For each status game it holds $\# E^{(N)} \leq 1$.* $\diamond$

Proof. We may suppose $E^{(N)} \neq \emptyset$. Let $\mathcal{C} \in \mathcal{P}_N$ be such that $E_{\mathcal{C}} \neq \emptyset$. Consider another partition $\mathcal{C}' \in \mathcal{P}_N$ with $\mathcal{C}' \neq \mathcal{C}$. The proof is complete if we show that $E_{\mathcal{C}'} = 0$, which will be done by contradiction.

So suppose $E_{\mathcal{C}'} \neq 0$. Let λ and λ' be the label functions of $\mathcal{C}$ and $\mathcal{C}'$. Because $\mathcal{C} \neq \mathcal{C}'$, by Lemma 45 in Appendix B (with $K = N$),[33] there exists $a, b \in \mathcal{N}$ such that

$$\lambda(a) < \lambda(b),\ \lambda'(a) = \lambda(b),\ \lambda'(b) < \lambda(b).$$

Proposition 23 with $k = \lambda(a)$ and $l = \lambda(b)$ implies

$$\bar{x}^a(q_{\lambda(b)}, q_{\lambda(a)}) \leq \hat{x}^b(q_{\lambda(b)})$$

and with $k = \lambda'(b)$ and $l = \lambda'(a)$ implies $\bar{x}^b(q_{\lambda'(a)}, q_{\lambda'(b)}) \leq \hat{x}^a(q_{\lambda'(a)})$, i.e.

$$\bar{x}^b(q_{\lambda(b)}, q_{\lambda'(b)}) \leq \hat{x}^a(q_{\lambda(b)}).$$

Noting that $q_{\lambda(a)} < q_{\lambda(b)}$ and $q_{\lambda'(b)} < q_{\lambda(b)}$ and using Lemma 4, we now obtain

$$\hat{x}^a(q_{\lambda(b)}) < \bar{x}^a(q_{\lambda(b)}, q_{\lambda(a)}) \leq \hat{x}^b(q_{\lambda(b)}) < \bar{x}^b(q_{\lambda(b)}, q_{\lambda'(b)}) \leq \hat{x}^a(q_{\lambda(b)}).$$

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[31] For suppose $\mathcal{C} \neq \mathcal{C}'$ and, to the contrary, $E_{\mathcal{C}} \cap E_{\mathcal{C}'} \neq \emptyset$. Then there exists $\mathbf{n}$ such that $\mathbf{n} \in E_{\mathcal{C}}$ and $\mathbf{n} \in E_{\mathcal{C}'}$. So both $\mathcal{C}$ and $\mathcal{C}'$ are cluster partitions of $\mathbf{n}$, which implies $\mathcal{C} = \mathcal{C}'$ – a contradiction.

[32] This result follows directly from our key Propositions 27 and 28 mentioned below, therefore, a proof is omitted. For clarity, Appendix B contains a proof that does not rely on these propositions.

[33] Lemma 45 shows that, given two different partitions of $\mathcal{P}_N$, we can always find a pair of players who are shifting places such that that one gets the higher cluster label of the other and the other gets some arbitrary lower cluster label.

This is a contradiction. ■

Using Lemma 4, it is not difficult to verify that the inequalities in Proposition 23 are met if the following more simple conditions hold:

$$\bar{x}^{\lambda^{-1}(k)}(s,0) < \hat{x}^{\lambda^{-1}(k+1)}(s) \quad (k \in \mathcal{N}\backslash\{N\}, s \in Q_{++}). \tag{18}$$

Since $\bar{x}^{\lambda^{-1}(k)}(s,0)$ is the maximum of $V_s^{\lambda^{-1}(k)}$ and $\hat{x}^{\lambda^{-1}(k+1)}(s)$ is the minimum of $V_s^{\lambda^{-1}(k+1)}$, these conditions are satisfied if for all pairs of heterogeneous players the non-degenerate V-sets are ordered disjoint sets put in the right order:

$$V_s^{\lambda^{-1}(1)} < \cdots < V_s^{\lambda^{-1}(N)} \quad (s \in Q_{++}). \tag{19}$$

It follows that, given partition $\mathcal{C}$, the fully diverse equilibrium exists if the differences among the ordered players are sufficiently large. Indeed, in that case, it is even the only Nash equilibrium:

Theorem 25 *Consider a status game with $M = N$ for which for all pairs of heterogeneous players the non-degenerate V-sets are ordered disjoint sets. Then*

- $E = E^{(N)}$,
- $\#E = 1$. ◇

Proof. By Lemma 21 there exists a unique ordered partition $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_N)$ of $\mathcal{N}$ where each $\mathcal{C}_k$ is a homogeneity class (with only one member now) such that, with λ the label function of $\mathcal{C}$, $V_s^{\lambda^{-1}(1)} < \cdots < V_s^{\lambda^{-1}(N)}$ $(s \in Q_{++})$. So (19) holds and, therefore, also the inequalities in Proposition 23. With Theorem 24, we have $\#E^{(N)} = 1$.

The proof is complete if we show that $E\backslash E^{(N)} = \emptyset$. We do this by contradiction. So suppose $\mathbf{n} \in E$ with $\mathbf{n} \notin E^{(N)}$ (so, because $M = N$, $\mathbf{n}$ is a clustering equilibrium). Then there exists a pair of players i and j for which $n := n^i = n^j$ and, by Proposition 9, $s := s^i(\mathbf{n}) = s^j(\mathbf{n}) > 0$. Applying Proposition 13, it follows that $n \in V_s^i$ and $n \in V_s^j$, so $V_s^i \cap V_s^j \neq \emptyset$. But (19) implies $V_s^i \cap V_s^j = \emptyset$, so we have a contradiction. ■

In sum, if the differences among players are so prominent that their non-degenerate V-sets are ordered disjoint sets, then there exists a unique Nash equilibrium. It is a fully diverse equilibrium where each player takes his optimal action $\hat{x}^i$ given a certain status level.

4.1.4 On the set of K-level equilibria $E^{(K)}$

Sofar we have dealt with two borderline cases: 1-level and N-level equilibria. In this section, we take up the general case and derive necessary and sufficient conditions for the existence of any type of Nash equilibrium.

First, recall that the players can be allocated to M homogeneity classes. From Theorem 15, we know a K-level equilibrium must be such that each of the K clusters is a union of the M classes. It follows that the number of equilibrium action levels is restricted by the number of classes: $K \leq M$.

Further, using Proposition 13, we find that if a K-level equilibrium exists, the action level of a cluster of players is an element of the intersection of their V-sets at the relevant status level. Defining, for a given ordered partition $(\mathcal{C}_1, \ldots, \mathcal{C}_K)$ of $\mathcal{N}$,

$$s_k := q_{\Sigma_{j=1}^k \#\mathcal{C}_j} \quad (k \in \mathcal{K} \cup \{0\}), \tag{20}$$

we have

Proposition 26 *Suppose* $\mathbf{n}$ *is a* K*-level Nash equilibrium. Then, with* $(\mathcal{C}_1, \ldots, \mathcal{C}_K)$ *the cluster partition and* $c_1 < \cdots < c_K$ *the level data of* $\mathbf{n}$*,*

$$c_k \in \cap_{i \in \mathcal{C}_k} V^i_{s_k} \quad (k \in \mathcal{K}). \quad \diamond$$

Proof. By Proposition 13, $n^i \in V^i_{s^i(\mathbf{n})}$. With λ the label function of the cluster partition, $s^i(\mathbf{n}) = q_{\Sigma_{j=1}^{\lambda(i)} \# \mathcal{C}_j}$ (Lemma 46 in Appendix B). Now take $k \in \mathcal{K}$ and $i \in \mathcal{C}_k$. Then $c_k = n^i \in V^i_{q_{\Sigma_{j=1}^{\lambda(i)} \# \mathcal{C}_j}} = V^i_{q_{\Sigma_{j=1}^{k} \# \mathcal{C}_j}} = V^i_{s_k}$ (using (20)), as desired. ■

The next rather technical but important proposition refines this result:

Proposition 27 *Suppose* $\mathbf{n}$ *is a* K*-level Nash equilibrium. Let* $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_K)$ *be the cluster partition and* $c_1 < \cdots < c_K$ *the level data of* $\mathbf{n}$*. Let* $\lambda : \mathcal{N} \to \mathcal{K}$ *be the label function of* $\mathcal{C}$*. With* $c_0 := 0$*, define for* $i \in \mathcal{N}$ *the numbers* y^i_j $(0 \leq j < \lambda(i))$ *by*

$$y^i_j := \max \left\{ \hat{x}^i(s_j + \frac{1}{N-1}), \min\{c_j, \bar{x}^i(s_j + \frac{1}{N-1}, 0)\} \right\}.$$

Then for each $k \in \mathcal{K}$

- $c_k \in \cap_{i \in \mathcal{C}_k} [\hat{x}^i(s_k), \min_{0 \leq j < k} \bar{x}^i_+(s_k, s_j + \frac{1}{N-1}, y^i_j)]$.[34]
  In particular,
  $$\#\mathcal{C}_k = 1 \Rightarrow c_k = \hat{x}^{\lambda^{-1}(k)}(s_k).$$
- $c_k \geq \bar{x}^i_+(s_k, s_j, c_j) \quad (1 \leq j < k,\ i \in \mathcal{C}_j)$.[35] $\quad \diamond$

(Because the proof fills a number of pages, and is moved to Appendix A.)

Proposition 27 implies that, in a K-level equilibrium, player i of cluster k has status level s_k. The first result shows that if player i is the only member of cluster k, and so is heterogeneous with respect to all other players, he chooses his optimal action $\hat{x}^i(s_k)$. If player i is not the only one,[36] he generally takes a higher action than $\hat{x}^i(s_k)$ that is an element of the intersection of certain individual member sets. The upper bound of player i's set (the min expression) indicates the action level that, if it yielded status s_k, would give player i the same payoff as his best action with lower status than s_k. Also note that this set is a subset of $V^i_{s_k}$,[37] so also the intersection is a subset of $\cap_{i \in \mathcal{C}_k} V^i_{s_k}$. The second result is a generalization of the inequality constraints in Proposition 23:[38] it implies that player i of cluster k cannot increase his payoff by choosing one of the the action levels with higher status than s_k.

Proposition 27 gives necessary conditions for a K-level equilibrium to exist. For each $k \in \{1, \ldots, K\}$, there are two conditions and both are on the cluster partition and the level data $c_1 < \cdots < c_K$ of $\mathbf{n}$. Are these conditions together also sufficient for the existence of a K-level equilibrium? The next proposition shows that this is indeed the case. Note that it is not clear, however, whether these conditions can always be satisfied, so its existence is not guaranteed.

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[34] Here $\bar{x}^i_+(s_k, s_j + \frac{1}{N-1}, y^i_j)$ is well-defined, because $y^i_j \in V^i_{s_j + 1/(N-1)}$.

[35] Here $\bar{x}^i_+(s_k, s_j, c_j)$ is well-defined, because $c_j \in V^i_{s_j}$.

[36] He may still be heterogeneous with respect to all other players.

[37] Because $\min_{0 \leq j < k} \bar{x}^i_+(s_k, s_j + \frac{1}{N-1}, y^i_j) \leq \bar{x}^i_+(s_k, s_{k-1} + \frac{1}{N-1}, y^i_{k-1}) \leq \bar{x}^i(s_{k-1} + \frac{1}{N-1}, 0) \leq \bar{x}^i(s_k, 0)$, which is the upper bound of $V^i_{s_k}$.

[38] This may be seen by rewriting the second result as $\bar{x}^i_+(s_l, s_k, c_k) \leq c_l$ $(k < l \leq K,\ i \in \mathcal{C}_k)$.

Proposition 28 *Fix an ordered partition* $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_K)$ *of* $\mathcal{N}$*. Suppose there exist* $c_1, \dots, c_K$ *such that for each* $k \in \mathcal{K}$

- $c_k \in \cap_{i \in \mathcal{C}_k} [\hat{x}^i(s_k), \min_{0 \le j < k} \bar{x}^i_+(s_k, s_j + \frac{1}{N-1}, y^i_j)]$,[39] *where, with* $c_0 := 0$,
$$y^i_j := \max \left\{ \hat{x}^i(s_j + \frac{1}{N-1}), \min\{c_j, \bar{x}^i(s_j + \frac{1}{N-1}, 0)\} \right\} \quad (i \in \mathcal{C}_k, 0 \le j < k),$$
- $c_k \ge \bar{x}^i_+(s_k, s_j, c_j) \quad (1 \le j < k,\, i \in \mathcal{C}_j)$.[40]

Then
$$c_1 < \dots < c_K$$
and, with $\mathbf{n}$ *the unique element of* $\mathbf{X}$ *that has* $\mathcal{C}$ *as cluster partition and* $c_1, \dots, c_K$ *as level data,* $\mathbf{n}$ *is a* K*-level Nash equilibrium.* $\diamond$

Proof. We start proving that $c_1 < \dots < c_K$. Take $k \in \mathcal{K}$ with $k < K$. The second statement with $l = k+1$ and $i \in \mathcal{C}_k$ gives $c_{k+1} \ge \bar{x}^i_+(s_{k+1}, s_k, c_k)$. From Lemma 5(*ii*), we know that $\bar{x}^i_+(s_{k+1}, s_k, c_k) > c_k$. Hence, $c_{k+1} > c_k$ ($k \in \mathcal{K} \backslash \{K\}$).

Now, let $\lambda : \mathcal{N} \to \mathcal{K}$ be the label function of $\mathcal{C}$. Then $n^i = c_{\lambda(i)}$ ($i \in \mathcal{N}$). Because $(\mathcal{C}_1, \dots, \mathcal{C}_K)$ is the cluster partition of $\mathbf{n}$, Lemma 46 in Appendix B implies $s^i(\mathbf{n}) = s_{\lambda(i)}$. Of course, $\mathbf{n}$ is a K-level multi-action – it remains to show that $\mathbf{n}$ is a Nash-equilibrium. Therefore, fix i and $x^i \in X^i$ with $x^i \neq c_{\lambda(i)}$. We must prove that
$$u^i(x^i, F_{\mathbf{n}^i}(x^i)) \le u^i(c_{\lambda(i)}, s_{\lambda(i)}).$$

- Case $x^i < c_{\lambda(i)}$. Let $k \in \{0, \dots, \lambda(i) - 1\}$ be such that $c_k \le x^i < c_{k+1}$. By Lemma 46, we have
$$F_{\mathbf{n}^i}(x^i) = s_k + \frac{1}{N-1}.$$
Because
$$c_k \le \hat{x}^i(s_k + \frac{1}{N-1}) \;\Rightarrow\; y^i_k = \hat{x}^i(s_k + \frac{1}{N-1}),$$
$$\hat{x}^i(s_k + \frac{1}{N-1}) < c_k < \bar{x}^i(s_k + \frac{1}{N-1}, 0) \;\Rightarrow\; y^i_k = c_k,$$
$$\bar{x}^i(s_k + \frac{1}{N-1}, 0) \le c_k \;\Rightarrow\; y^i_k = \bar{x}^i(s_k + \frac{1}{N-1}, 0),$$
we have
$$u^i(x^i, s_k + \frac{1}{N-1}) \le u^i(y^i_k, s_k + \frac{1}{N-1}).$$
Using this and noting that $y^i_k \in V^i_{s_k + \frac{1}{N-1}}$, we obtain
$$u^i(x^i, F_{\mathbf{n}^i}(x^i)) = u^i(x^i, s_k + \frac{1}{N-1})$$
$$\le u^i(y^i_k, s_k + \frac{1}{N-1}) = u^i(\bar{x}^i_+(s_{\lambda(i)}, s_k + \frac{1}{N-1}, y^i_k), s_{\lambda(i)})$$
$$\le u^i(\min_{0 \le l < \lambda(i)} \bar{x}^i_+(s_{\lambda(i)}, s_l + \frac{1}{N-1}, y^i_l), s_{\lambda(i)}) \le u^i(c_{\lambda(i)}, s_{\lambda(i)}).$$

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[39] See note 34. As before, if $\#\mathcal{C}_k = 1$, say $\mathcal{C}_k = \{l\}$, then $c_k = \hat{x}^l(s_k)$ (since $[\hat{x}^l(s_k), \min_{0 \le j < k} \bar{x}^l_+(s_k, s_j + \frac{1}{N-1}, y^l_j)] = \{\hat{x}^l(s_k)\}$ – see the proof of Proposition 27).

[40] See note 35.

- Case $x^i > c_{\lambda(i)}$. Let $k \in \{\lambda(i), \dots, K\}$ be such that, with $c_{K+1} := \infty$, $c_k \le x^i < c_{k+1}$. By Lemma 46, we have
$$F_{\mathbf{n}^{\hat{\imath}}}(x^i) = s_k.$$
Also, by the second statement,
$$\hat{x}^i(s_k) \le \bar{x}^i_+(s_k, s_{\lambda(i)}, c_{\lambda(i)}) \le c_k \le x^i.$$
We now obtain
$$u^i(x^i, F_{\mathbf{n}^{\hat{\imath}}}(x^i)) = u^i(x^i, s_k) \le u^i(\bar{x}^i_+(s_k, s_{\lambda(i)}, c_{\lambda(i)}), s_k) = u^i(c_{\lambda(i)}, s_{\lambda(i)}).$$

This completes our proof. ■

An example may illustrate the gist of Propositions 27 and 28 (our calculations use the outcomes of example 7):

Example 29 *Consider a status game with three players, so* $\mathcal{N} = \{1, 2, 3\}$. *As in Example 7, suppose the payoff function of player* i *is* $u^i(x, s) := - \mid x - A^i \mid + B^i s + C^i$, *where* $A^i, B^i, C^i > 0$ *and* $L^i > A^i + B^i$ $(i \in \mathcal{N})$. *Assume* $A^1 < A^2 < A^3$ *and* $B := B^1 = B^2 = B^3$. *The next results follow from Propositions 27 and 28 after some calculation (see also Appendix A):*

- *Suppose* $\mathcal{C} = (\mathcal{C}_1) = (\{1, 2, 3\})$. *Then the multi-action* $\mathbf{n}$ *with* $n^i := c_1$ *for* $i \in \mathcal{C}_1$ *is a Nash equilibrium if and only if* $c_1 \in \cap_{i \in \mathcal{C}_1}[A^i, A^i + B]$. *Hence, a 1-level equilibrium exists if and only if* $A^1 + B \ge A^3$.

- *Suppose* $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = (\{1\}, \{2\}, \{3\})$. *Then the multi-action* $\mathbf{n}$ *with* $n^i := c_k$ *for* $i \in \mathcal{C}_k$ $(k \in \{1, 2, 3\})$ *is a Nash equilibrium if and only if* $c_k = A^i$ *and* $A^1 + B \le A^2 + \frac{1}{2}B \le A^3$, *and it is a 3-level equilibrium.*

- *Suppose* $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) = (\{1, 2\}, \{3\})$. *Then the multi-action* $\mathbf{n}$ *with* $n^i := c_1$ *for* $i \in \mathcal{C}_1$ *and* $n^i := c_2$ *for* $i \in \mathcal{C}_2$ *is a Nash equilibrium if and only if* $c_1 \in \cap_{i \in \mathcal{C}_1}[A^i, A^i + \frac{1}{2}B]$, $c_2 = A^3$, *and* $c_1 \le A^3 - \frac{1}{2}B$. *Hence, given this ordered partition, a 2-level equilibrium exists if* $A^2 + \frac{1}{2}B \le A^1 + B \le A^3$.

- *Suppose* $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) = (\{1\}, \{2, 3\})$. *Then the multi-action* $\mathbf{n}$ *with* $n^i := c_1$ *for* $i \in \mathcal{C}_1$ *and* $n^i := c_2$ *for* $i \in \mathcal{C}_2$ *is a Nash equilibrium if and only if* $c_1 = A^1$, $c_2 \in \cap_{i \in \mathcal{C}_2}[A^i, A^i + \frac{1}{2}B]$, *and* $c_2 \ge A^1 + B$. *Hence, given this ordered partition, a 2-level equilibrium exists if* $A^1 + B \le A^3 \le A^2 + \frac{1}{2}B$.

Using the last two results, it is easily verified that none of the other four possible ordered partitions gives rise to a 2-level equilibrium. ◇

In the preceding example, we have identified all Nash equilibria. Comparing the inequality conditions of the four types, we see that a Nash equilibrium always exists. It is even possible that all types of Nash equilibria exist simultaneously (viz., if $A^1 + B = A^2 + \frac{1}{2}B = A^3$), so we may have different types of clustering equilibria alongside a single-level equilibrium and a fully diverse equilibrium. This special case also implies that if $\mathcal{C}$ is a K-dimensional ordered partition for which a K-level equilibrium exists, there may be another K-dimensional ordered partition $\mathcal{C}'$ for which a K-level equilibrium exists as well. But recall from Theorem 24 that this is not possible if $K = N$.

4.1.5 On the set of fully diverse equilibria $E^{(M)}$

In Section 4.1.3, we have studied the set of fully diverse equilibria for the case of complete heterogeneity. One conclusion was that if the differences among players are large enough, there exists a unique Nash equilibrium and, in this equilibrium, all players take different actions. We will now continue our investigation of fully diverse equilibria for the general case of M homogeneity classes. Some results follow from the technical Propositions 27 and 28 above.

We simplify our notation a bit. For $M \in \{1, \ldots, N\}$, define $\mathcal{M} := \{1, \ldots, M\}$. Suppose $\mathcal{C}_k$ is a homogeneity class. Then, from Proposition 12 and Lemma 6, for each $k \in \mathcal{M}$ and $i, j \in \mathcal{C}_k$ the functions $\hat{x}^i$ and $\hat{x}^j$ are the same and also the functions $\bar{x}^i$ and $\bar{x}^j$, so we can define $\hat{x}_k := \hat{x}^i = \hat{x}^j$ and $\bar{x}_k := \bar{x}^i = \bar{x}^j$.

Proposition 30 *Let $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_M)$ be an ordered partition of $\mathcal{N}$ such that each $\mathcal{C}_k$ is a homogeneity class. If, for all $k = 1, \ldots, M-1$, the strict inequalities*

$$\bar{x}_k(s, 0) < \hat{x}_{k+1}(s) \quad (s \in Q_{++}) \tag{21}$$

hold, then

(1) $E_{\mathcal{C}} \neq \emptyset$, so there exists a fully diverse equilibrium. Even, $E_{\mathcal{C}}$ is equal to the set of multi-actions with cluster partition $\mathcal{C}$ and level data $c_1 < \cdots < c_M$ such that

$$c_k \in [\hat{x}_k(s_k), \bar{x}_k(s_k, s_{k-1} + \frac{1}{N-1})] \quad (k \in \mathcal{M}).$$

(2) If $M < N$, then $\# E_{\mathcal{C}} = \infty$.

(3) For each ordered partition $\mathcal{C}' = (\mathcal{C}'_1, \ldots, \mathcal{C}'_M)$ of $\mathcal{N}$ with $\mathcal{C}' \neq \mathcal{C}$ such that each $\mathcal{C}'_k$ is a homogeneity class, it holds $E_{\mathcal{C}'} = \emptyset$.

(4) $E^{(K)} = \emptyset$ for all $K < M$, so there does not exist a clustering equilibrium. $\diamond$

Proof. (1) This result follows from Proposition 28, as shown in Appendix A.

(2) Suppose $M < N$. Then there exists $k \in \mathcal{M}$ such that $\# \mathcal{C}_k \geq 2$. For such a k, it holds $s_k < s_{k-1} + \frac{1}{N-1}$ and, therefore, $\hat{x}_k(s_k) < \bar{x}_k(s_k, s_{k-1} + \frac{1}{N-1})$. Now apply result (1).

(3) By contradiction. So suppose $E_{\mathcal{C}'} \neq \emptyset$. Let λ and λ' be the label functions of $\mathcal{C}$ and $\mathcal{C}'$. Applying Lemma 45 to $\mathcal{C}$ and $\mathcal{C}'$ with $K = M$, there exist $i, j \in \mathcal{N}$ with

$$\lambda(i) < \lambda(j),\ \lambda'(i) = \lambda(j),\ \lambda'(j) < \lambda(j).$$

Define $k := \lambda(i)$, $l := \lambda(j)$, and $r := \lambda'(j)$. Using accents for the relevant notations for $\mathcal{C}'$ and noting that $\mathcal{C}'_l = \mathcal{C}_k$ and $\mathcal{C}'_r = \mathcal{C}_l$, we have

$$\bar{x}'_{+l} = \bar{x}_{+k},\ \bar{x}'_{+r} = \bar{x}_{+l}$$

(similar outcomes for $\hat{x}$ and $\bar{x}$). Now, let $c'_1 < \cdots < c'_M$ be the level data of some element of $E_{\mathcal{C}'}$. Then, according to result (1),

$$c'_r \in [\hat{x}_l(s'_r), \bar{x}_l(s'_r, s'_{r-1} + \frac{1}{N-1})],\ c'_l \in [\hat{x}_k(s'_l), \bar{x}_k(s'_l, s'_{l-1} + \frac{1}{N-1})].$$

Moreover, from Proposition 27,

$$c'_l \geq \bar{x}_{+l}(s'_l, s'_r, c'_r).$$

It follows that

$$c'_l \geq \bar{x}_{+l}(s'_l, s'_r, c'_r) \geq \bar{x}_{+l}(s'_l, s'_r, \hat{x}_l(s'_r)) = \bar{x}_l(s'_l, s'_r) > \hat{x}_l(s'_l).$$

But, since $\bar{x}_k(s,0) < \hat{x}_l(s) < \bar{x}_l(s,0)$ for $s \in Q_{++}$ by assumption and noting that $s'_l > 0$, we also have

$$c'_l \leq \bar{x}_k(s'_l, s'_{l-1} + \frac{1}{N-1}) \leq \bar{x}_k(s'_l, 0) < \hat{x}_l(s'_l).$$

This is a contradiction.

(4) By contradiction. So suppose $\mathbf{n}$ is a clustering equilibrium. Then there exist homogeneity classes $\mathcal{C}_k$ and $\mathcal{C}_l$ with $k < l$ such that all their members have the same action level in $\mathbf{n}$. Let $i \in \mathcal{C}_k$ and $j \in \mathcal{C}_l$. So we can define $n := n^i = n^j$ and, therefore, $s := s^i(\mathbf{n}) = s^j(\mathbf{n}) > 0$. By Proposition 13, we have $n \in V^i_s = [\hat{x}_k(s), \bar{x}_k(s,0)]$ and $n \in V^j_s = [\hat{x}_l(s), \bar{x}_l(s,0)]$, so $V^i_s \cap V^j_s \neq \emptyset$. But (21) implies $\bar{x}_k(s,0) < \hat{x}_l(s)$, so $V^i_s \cap V^j_s = \emptyset$ – a contradiction. $\blacksquare$

Proposition 30 assumes that there is a certain "minimum distance" between the homogeneity classes. Condition (21) is similar to condition (18), so we know that it is satisfied if for all pairs of heterogeneous players the non-degenerate V-sets are ordered disjoint sets put in the right order. In the presence of homogeneity classes with two or more members (so $M < N$), there exists a continuum of fully diverse equilibria, because the equilibrium action of such a homogeneity class belongs to an interval. Moreover, this family of fully diverse equilibria is unique in the sense that no fully diverse equilibria exist under alternative rankings of the homogeneity classes. The distance between homogeneity classes also rules out the possibility of clustering equilibria. The next theorem summarizes some basic equilibrium properties for the case of sufficiently heterogeneous players (cf. Theorem 25):

Theorem 31 *Consider a status game with $M < N$ for which for all pairs of heterogeneous players the non-degenerate V-sets are ordered disjoint sets. Then*

- $E = E^{(M)}$,
- $\#E = \infty$. $\diamond$

Proof. By Lemma 21 there exists a unique ordered partition $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_M)$ of $\mathcal{N}$ where each $\mathcal{C}_k$ is a homogeneity class, such that $V_{1s} < \cdots < V_{Ms}$ $(s \in Q_{++})$.[41] Proposition 30 applies to this partition. Its result (4) implies '$E = E^{(M)}$' and, because $E_{\mathcal{C}} \subseteq E^{(M)}$, its result (2) implies '$\#E = \infty$'. $\blacksquare$

4.2 Pareto efficiency of Nash equilibria

We now turn to the efficiency question. Although it is straightforward enough to show that each status game has a Pareto-efficient multi-action,[42] the efficiency properties of Nash equilibria are rather difficult to disclose. For the case of only two (heterogeneous) players, we already found that fully diverse equilibria are always Pareto efficient whereas clustering equilibria are generally not, although there always exists an efficient one (see Section 3.3). For more than two players, however, this conclusion must be modified. Our main findings are summarized by Corollaries 33 and 35 and Proposition 36, which are largely based on two key results.

Here is our first key result (recall $s_k := q_{\Sigma_{j=1}^k \# \mathcal{C}_j}$):

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[41] Here V_{ks} denotes the (identical) sets V^i_s $(i \in \mathcal{C}_k)$.

[42] Each maximizer of $W := \Sigma_{i=1}^N v^i$ is Pareto efficient. To see that W has indeed a maximizer, note that, by Proposition 10, v^i is upper semi-continuous on the non-empty compact set $\mathbf{X}$, so the same holds for W. Applying the Weierstrass-Lebesque lemma, therefore, W has a maximizer.

Proposition 32 *Suppose* $\mathbf{n}$ *is a* K*-level Nash equilibrium with cluster partition* $(\mathcal{C}_1, \ldots, \mathcal{C}_K)$ *and level data* $c_1 < \cdots < c_K$ *such that*

$$c_k = \hat{x}^i(s_k) \quad (k \in \mathcal{K},\ i \in \mathcal{C}_k).$$

Then $\mathbf{n}$ *is Pareto efficient.* $\diamond$

Proof. With $\lambda : \mathcal{N} \to \mathcal{K}$ the label function of $(\mathcal{C}_1, \ldots, \mathcal{C}_K)$, we have for all $i \in \mathcal{N}$

$$n^i = c_{\lambda(i)} = \hat{x}^i(s_{\lambda(i)}) \quad \text{and } s^i(\mathbf{n}) = s_{\lambda(i)}. \tag{22}$$

Because $\mathbf{n}$ is a Nash equilibrium, Proposition 27 can be applied. So we know that, for each $k \in \mathcal{K}$, $c_k \geq \bar{x}^i_+(s_k, s_j, c_j)$ $(1 \leq j < k,\ i \in \mathcal{C}_j)$. Noting that $\bar{x}^i_+(s_k, s_j, c_j) = \bar{x}^i_+(s_k, s_j, \hat{x}^i(s_j)) = \bar{x}^i(s_k, s_j)$, we have for each $k \in \mathcal{K}, 1 \leq j < k,$ and $i \in \mathcal{C}_j$

$$c_k \geq \bar{x}^i_+(s_k, s_j). \tag{23}$$

Now suppose, without loss of generality, that the label function λ is increasing. Let $\mathbf{a} \in \mathbf{X}$ be a multi-action such that for all $i \in \mathcal{N}$

$$v^i(\mathbf{a}) \geq v^i(\mathbf{n}).$$

Then, noting (22), we can apply Lemma 50 in Appendix B. The proof is complete if we can show that, for all $i \in \mathcal{N}$, $v^i(\mathbf{a}) \leq v^i(\mathbf{n})$.

According to Lemma 50(2), we have two cases for $i \in \mathcal{N}$:

- Case $a^i = n^i$ and $s^i(\mathbf{a}) = s_{\lambda(i)}$. Then (even)

$$v^i(\mathbf{a}) = u^i(a^i, s^i(\mathbf{a})) = u^i(n^i, s_{\lambda(i)}) = v^i(\mathbf{n}).$$

- Case $s^i(\mathbf{a}) > s_{\lambda(i)}$. Then $i \leq N - 1$. Noting that, by Lemma 50(1), $s^N(\mathbf{a}) = s_{\lambda(N)}$, let $p \geq i$ with $p < N$ be such that $s^l(\mathbf{a}) > s_{\lambda(l)}$ $(i \leq l \leq p)$ and $s^{p+1}(\mathbf{a}) = s_{\lambda(p+1)}$. Then Lemma 50(4) implies

$$a^i = a^{i+1} = \cdots = a^{p+1},$$

so

$$s^i(\mathbf{a}) = s_{\lambda(p+1)}.$$

With (23) and Lemmas 4 and 50(6), it follows that

$$\hat{x}^i(s_{\lambda(p+1)}) \leq \bar{x}^i(s_{\lambda(p+1)}, s_{\lambda(i)}) \leq c_{\lambda(p+1)} = n^{p+1} \leq a^{p+1}.$$

Using this, we arrive at

$$\begin{aligned} v^i(\mathbf{a}) &= u^i(a^i, s^i(\mathbf{a})) = u^i(a^{p+1}, s_{\lambda(p+1)}) \\ &\leq u^i(\bar{x}^i(s_{\lambda(p+1)}, s_{\lambda(i)}), s_{\lambda(p+1)}) \\ &= u^i(\hat{x}^i(s_{\lambda(i)}), s_{\lambda(i)}) = u^i(n^i, s^i(\mathbf{n})) = v^i(\mathbf{n}). \end{aligned}$$

So our proof is complete. ■

Proposition 32 provides the intuitive result that if we have a Nash equilibrium with K status levels where each player i takes his optimal action $\hat{x}^i$ at his acquired status level s_k $(k = 1, \ldots, K)$, then the equilibrium is Pareto efficient. It implies that in case of complete heterogeneity, so if $M = N$, each fully diverse equilibrium is Pareto efficient:

Corollary 33 *If $M = N$, each fully diverse equilibrium of a status game is Pareto efficient.* $\diamond$

Proof. Suppose $\mathbf{n}$ is a fully diverse equilibrium, so $\mathbf{n}$ is a N-level equilibrium. Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_N)$ be its cluster partition and $c_1 < \cdots < c_N$ its level data. Proposition 21 implies

$$c_k = \hat{x}^i(q_k) \ (k \in \mathcal{N}, i \in \mathcal{C}_k).$$

Note that, because $\mathcal{C} \in \mathcal{P}_N$, it holds $s^i(\mathbf{n}) = q_k$ $(i \in \mathcal{C}_k)$ and so $s_k = q_k$. $\blacksquare$

Under complete heterogeneity, a fully diverse equilibrium exists if the differences among players are sufficiently large (see Proposition 23), and now we know that it is also Pareto efficient. Remember that if these differences are even such that the V-sets of the players are ordered disjoint sets,[43] the fully diverse equilibrium is the only Nash equilibrium that exists (see Theorem 25).

The question arises whether each fully diverse equilibrium is also Pareto efficient in the absence of complete heterogeneity, so when two or more players are homogeneous. In this case, any ordered partition of the players includes a cluster with two or more members, which makes that a fully diverse equilibrium is generally not unique. To evaluate the possibility of multiple Nash equilibria, we have the following key result.

Proposition 34 *Suppose $\mathbf{d}$ is a K-level multi-action such that, with cluster partition $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_K)$ and level data $c_1 < \cdots < c_K$,*

$$c_k \geq \max_{i \in \mathcal{C}_k} \hat{x}^i(s_k) \quad (k \in \mathcal{K}). \tag{24}$$

Then for all $\mathbf{x}$ with cluster partition $\mathcal{C}$ and with $\mathbf{x} > \mathbf{d}$, $\mathbf{d}$ is a Pareto improvement of $\mathbf{x}$. $\diamond$

Proof. Denote by $\lambda : \mathcal{N} \to \mathcal{K}$ the label function of $\mathcal{C}$. Let $\mathbf{c} = (c_1, \dots, c_K)$ be the level data of $\mathbf{d}$ and $\mathbf{c}' = (c'_1, \dots, c'_K)$ those of $\mathbf{x}$. Because $\mathbf{x} > \mathbf{d}$, it holds $\mathbf{c}' > \mathbf{c}$. Because both $\mathbf{d}$ and $\mathbf{x}$ have cluster partition $\mathcal{C}$, $s^i(\mathbf{d}) = s^i(\mathbf{x}) = s_{\lambda(i)}$. So we have

$$v^i(\mathbf{x}) = u^i(c'_{\lambda(i)}, s_{\lambda(i)}) \text{ and } v^i(\mathbf{d}) = u^i(c_{\lambda(i)}, s_{\lambda(i)}).$$

Noting that

$$c'_{\lambda(i)} \geq c_{\lambda(i)} \geq \max_{j \in \mathcal{C}_{\lambda(i)}} \hat{x}^j(s_{\lambda(i)}) \geq \hat{x}^i(s_{\lambda(i)}),$$

the inequalities $v^i(\mathbf{x}) \leq v^i(\mathbf{d})$ $(i \in \mathcal{N})$ follow. Now, since $\mathbf{c}' > \mathbf{c}$, let $l \in \mathcal{K}$ be such that $c'_l > c_l$. Then for each i with $\lambda(i) = l$, we have $c'_{\lambda(i)} > c_{\lambda(i)} \geq \hat{x}^i(s_{\lambda(i)})$ and, therefore, $v^i(\mathbf{x}) < v^i(\mathbf{d})$. This implies the desired result. $\blacksquare$

Note that each K-level Nash equilibrium satisfies the requirements of Proposition 34, as can be seen by inspecting the intersection expression in Proposition 27 (see Section 4.1.4). Hence, Proposition 34 can be used to evaluate fully diverse and clustering equilibria. Also note that if each cluster $\mathcal{C}_k$ is a homogeneity class, the inequalities (24) simply become $c_k \geq \hat{x}_k(s_k)$ $(k \in \mathcal{K})$.

We can now answer the efficiency question for fully diverse equilibria.

Corollary 35 *(fully diverse equilibria) Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_M)$ be an ordered partition of $\mathcal{N}$. Suppose $\mathbf{x}^*$ is a M-level multi-action with cluster partition $\mathcal{C}$ and level data $\max_{i \in \mathcal{C}_1} \hat{x}^i(s_1) < \cdots < \max_{i \in \mathcal{C}_M} \hat{x}^i(s_M)$. Then*

- *for all $\mathbf{n}, \mathbf{n}' \in E_{\mathcal{C}}$ with $\mathbf{n}' > \mathbf{n}$, $\mathbf{n}$ is a Pareto improvement of $\mathbf{n}'$;*
- *each $\mathbf{n} \in E_{\mathcal{C}}$ with $\mathbf{n} \neq \mathbf{x}^*$ is not Pareto efficient;*

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[43] For the terminology, see Section 4.1.2.

- *if* $\mathbf{x}^* \in E_{\mathcal{C}}$, $\mathbf{x}^*$ *is Pareto efficient.* $\diamond$

Proof. The first result follows from Proposition 34, as does the second result by noting that, with $\mathbf{d} = \mathbf{x}^*$, if $\mathbf{n} \neq \mathbf{d}$ then $\mathbf{n} > \mathbf{d}$. The third result follows from Proposition 32. $\blacksquare$

To fix ideas, suppose the homogeneity classes are sufficiently separated, as in Proposition 30 (see Section 4.1.5). Then there exists a unique family of fully diverse equilibria, and Corollary 35 states that these equilibria are generally inefficient. The inefficiency arises because the members of each homogeneity class typically overact: they choose an action level that is higher than their optimal action, $\hat{x}_k(s_k)$ if they belong to cluster k. A somewhat lower action level would yield a Pareto improvement and lead to a superior equilibrium. Contrary to the sub-optimality results in the literature, however, there does exist a Pareto-efficient Nash equilibrium, where indeed each player takes his optimal action. But note that the efficient equilibrium may not exist if the gaps between the homogeneity classes are less wide.

For clustering equilibria, we find a similar result, but with an important exception:

Proposition 36 *(Clustering equilibria) Suppose* $K < M$*. Let* $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_K)$ *be an ordered partition of* $\mathcal{N}$*. Suppose* $\mathbf{x}^*$ *is a* K*-level multi-action with cluster partition* $\mathcal{C}$ *and level data* $\max_{i \in \mathcal{C}_1} \hat{x}^i(s_1) < \cdots < \max_{i \in \mathcal{C}_K} \hat{x}^i(s_K)$*. Then*

- *for all* $\mathbf{n}, \mathbf{n}' \in E_{\mathcal{C}}$ *with* $\mathbf{n}' > \mathbf{n}$, $\mathbf{n}$ *is a Pareto improvement of* $\mathbf{n}'$*;*
- *each* $\mathbf{n} \in E_{\mathcal{C}}$ *with* $\mathbf{n} \neq \mathbf{x}^*$ *is not Pareto efficient;*
- *if* $\mathbf{x}^* \in E_{\mathcal{C}}$, $\mathbf{x}^*$ *is weakly Pareto efficient, but may not be Pareto efficient.* $\diamond$

Proof. The first result follows from Proposition 34, as does the second result by noting that, with $\mathbf{d} = \mathbf{x}^*$, if $\mathbf{n} \neq \mathbf{d}$ then $\mathbf{n} > \mathbf{d}$. Regarding the third result, take $l \in \mathcal{C}_K$ such that $c_K := \max_{i \in \mathcal{C}_K} \hat{x}^i(s_K) = \hat{x}^l(s_K)$ $(= \hat{x}^l(1))$. Then $x^{*l} = \hat{x}^l(1)$ and $x^{*l} \geq x^{*j}$ $(j \in \mathcal{N})$. So $v^l(\mathbf{x}^*) = u^l(\hat{x}^l(1), 1)$, which implies that $\mathbf{x}^*$ is weakly Pareto efficient. That $\mathbf{x}^*$ may not be strongly efficient is discussed below. $\blacksquare$

In general, a clustering equilibrium is not unique, but belongs to a continuum of clustering equilibria with the same cluster partition. As before, one member of this continuum is a Pareto improvement of another member if it has lower action levels. The difference with fully diverse equilibria arises when the multi-action $\mathbf{x}^*$ defined in Proposition 36 is a Nash equilibrium. Although then $\mathbf{x}^*$ Pareto dominates all the other clustering equilibria and is also weakly Pareto efficient,[44] it may be Pareto inefficient. Of course, if the optimal action levels are the same within each cluster, so if $\hat{x}^i(s_k)$ does not depend on i for each $k \in \mathcal{K}$, then $\mathbf{x}^*$ is Pareto efficient. But this is really a borderline case since, by definition, there is always a cluster with two or more homogeneity classes, and it is typical for these classes to have different $\hat{x}^i$ functions. Although it is true that different $\hat{x}^i$ functions within a cluster does not necessarily mean that $\mathbf{x}^*$ is inefficient,[45] this possibility cannot be ruled out, because heterogeneous players who share the same action level tend to be better off by separating their actions.

For illustration, suppose the cluster partition of $\mathbf{x}^*$ includes a cluster $p \in \mathcal{K}$ with two or more homogeneity classes, among which $\mathcal{H}$. With $c_1 < \cdots < c_K$ the level data of $\mathbf{x}^*$, suppose the players of $\mathcal{H}$ lower their action level from c_p to c^* such that $c_{p-1} < c^* < c_p$. So we have an alternative multi-action $\mathbf{x}$ with $K+1$ clusters where the members of $\mathcal{H}$ get a lower status $s^* := s_{p-1} + \frac{\#\mathcal{H}}{N-1} < s_p$ and all the other players remain at their status levels. Despite his

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[44] So there does not exist a multi-action that makes all players strictly better off than under $\mathbf{x}^*$.

[45] An example is provided by the two-players status game in Section 3.3.

lower status, each player $i \in \mathcal{H}$ can increase his payoff by choosing $c^* = \hat{x}^i(s^*)$, provided that $c_{p-1} < \hat{x}^i(s^*)$ and $\bar{x}^i(s_p, s^*) < c_p$.[46] Indeed, we then have for each $i \in \mathcal{H}$

$$u^i(c^*, s^*) = u^i(\hat{x}^i(s^*), s^*) = u^i(\bar{x}^i(s_p, s^*), s_p) > u^i(c_p, s_p).$$

Since the payoffs to all other players stay the same, therefore, the Nash equilibrium $\mathbf{x}^*$ is not Pareto efficient.

The principle that, in such a Pareto-dominating clustering equilibrium as $\mathbf{x}^*$, heterogeneous players with the same action level may do better by separating their actions is also illustrated in the next example.

Example 37 *In Example 29, we considered a status game with three players and established, among other things, the existence of a family of single-level equilibria. The Pareto-dominating member of this family is* $\mathbf{n} = (A^3, A^3, A^3)$*. The payoff to player* $i \in \{1, 2, 3\}$ *amounts to* $u^i(A^3, 1)$*.*

Now consider another multi-action where players 1 *and* 2 *take action* A^2 *and player* 3 *stays with* A^3*. Since* $A^2 < A^3$*, the payoff to player* 3 *remains the same. The payoffs to players* 1 *and* 2 *are* $u^1(A^2, \frac{1}{2})$ *and* $u^2(A^2, \frac{1}{2})$*, and the changes in payoffs*

$$u^1(A^2, \frac{1}{2}) - u^1(A^3, 1) = u^2(A^2, \frac{1}{2}) - u^2(A^3, 1) = A^3 - (A^2 + \frac{1}{2}B).$$

So the multi-action is a Pareto improvement if and only if $A^2 + \frac{1}{2}B < A^3$*. To see whether it can be satisfied, recall from Example 27 the equilibrium condition* $A^1 + B \geq A^3$*, which implies the restriction* $A^2 + B > A^3$*. Thus, by choosing the parameters such that* $A^1 < A^2 < A^1 + \frac{1}{2}B < A^2 + \frac{1}{2}B < A^3 \leq A^1 + B < A^2 + B$*, we find that the Pareto-dominating equilibrium above is not Pareto efficient.* ◇

5 Does status seeking reduce saving?

In this section, we illustrate the foregoing with a brief discussion of the impact of status seeking on savings behaviour. Frank (1985a, Ch. 8; 1985b, pp. 103-106) argues that, in their quest for a higher social position, consumers demand more status-generating or positional goods and fewer non-positional goods, as compared with a situation where they cannot alter their status. Because "savings" may be regarded as a non-positional good, status seeking thus reduces the consumer's average propensity to save.[47] Moreover, this effect is strongest for low-income consumers, so saving rates are not only lower across all income levels but also fall when we move to the lower tail of the income distribution. The latter agrees with the observed positive correlation between saving rates and income – something that is hard to reconcile with the Life-Cycle/Permanent-Income Hypothesis (cf. Dynan et al., 2004).

To verify these two effects on saving rates, consider the two-goods game Γ formulated in Section 2.2. Let us think of an intertemporal two-period setting with only income in period one, where $p_y y^i$ indicates (the present value of) second-period consumption and thus the amount saved, and $p_y y^i / w^i$ or $1 - p_x x^i / w^i$ refers to consumer i's saving rate. The exposition gains much clarity with a little more structure. Assume that the two-goods game is such that the associated status game Γ' meets the following two plausible conditions:

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[46] Since $\hat{x}^i(s_p) < \bar{x}^i(s_p, s^*)$, the latter condition requires $\hat{x}^i(s_p) < c_p$, which is satisfied by assumption for $i \in \mathcal{H}$: $\hat{x}^i(s_p) < c_p = \max_{j \in \mathcal{C}_p} \hat{x}^j(s_p)$.

[47] Lower saving rates do not only reflect the negative externalities of status seeking, but also slow down economic growth (see also Frank, 1997).

Condition 38
(i) for each $i \in \mathcal{N}$ *and* $s, s' \in Q : s < s' \Rightarrow \hat{x}^i(s) \leq \hat{x}^i(s')$;
(ii) for each $s \in Q$ *and* $i, j \in \mathcal{N} : w^i < w^j \Rightarrow \hat{x}^i(s) < \hat{x}^j(s)$. $\diamond$

Let $W_s^{ij} := (V_s^i \cup V_s^j) \cap (X^i \cap X^j)$ $(i, j \in \mathcal{N}, s \in Q)$.

Condition 39 *For all* $i, j \in \mathcal{N}$ *with* $w^i \leq w^j$, $s, s' \in Q$ *with* $s' \geq s$, *and* $x \in W_s^{ij}$ *and* $x' \in W_{s'}^{ij}$ *with* $x' > x$:

$$u^i(x', s') - u^i(x, s) > 0 \Rightarrow u^j(x', s') - u^j(x, s) > 0. \quad \diamond$$

In Section 2.2, we showed that $\hat{x}^i(s) = \hat{x}(s, w^i)$. So the first condition just says that the quantity demanded of first-period consumption is increasing in status and strictly increasing in income. Consider that, since present consumption is the positional good, it is natural to assume that higher status itself increases the demand for present consumption. The positive dependence on income indicates that present consumption is assumed to be a normal good.

The second condition stipulates a certain monotonicity. The condition is very powerful, as signified by Proposition 41 below. It basically states that if, over some common range of actions, the change in payoffs from a higher action (given the actions of the others) is positive for a specific consumer, then it is also positive for any other consumer whose income is not lower. Proposition 52 in Appendix B says that the condition holds if the underlying two-goods game has the following additional properties (recall that $u^i(x^i, s^i) := U(x^i, (w^i - p_x x^i)/p_y; s^i)$): the function U is twice continuously differentiable with partial derivatives $U_{11}, U_{22} \leq 0$ and $U_{12}, U_{23} \geq 0$. With this result it is not difficult to identify a general class of utility functions U that satisfy both conditions 38 and 39.[48]

Now, it follows that our M homogeneity classes are *income classes*:

Proposition 40 *Consider the two-goods game* Γ *and the associated status game* Γ'. *Then two players* $i, j \in \mathcal{N}$ *are homogeneous if and only if* $w^i = w^j$. $\diamond$

Proof. "If" is obvious. As for "only if", suppose i and j are homogeneous. Then $\hat{x}^i(0) = \hat{x}^j(0)$. Imposing condition 38(*ii*), this requires $w^i = w^j$. $\blacksquare$

Suppose the income vector $\mathbf{w}$ has level data $w_1 < \cdots < w_M$, and define, for each $k = 1, \ldots, M$, $\mathcal{H}_k := \{i \in \mathcal{N} \mid w^i = w_k\}$. So $\mathcal{H}_1$ refers to the lowest income class, $\mathcal{H}_2$ to the second lowest, and so on, until $\mathcal{H}_M$ follows as the highest income class. The next proposition shows that, in each Nash equilibrium, the distribution of present consumption is positively related to the income distribution. That is, in a fully diverse equilibrium, there are M quantities of present consumption, and higher consumption levels correspond to higher income classes. In a clustering equilibrium, there are fewer levels of present consumption, and at least one quantity of consumption is chosen by two or more successive income classes.

Proposition 41 *Consider the two-goods game* Γ *and the associated status game* Γ'. *Then for each Nash equilibrium* $\mathbf{n}$ *of game* Γ' *it holds*

- $n^i < n^j \Rightarrow w^i < w^j$;
- if $\mathbf{n}$ is a fully diverse equilibrium, $w^i < w^j \Leftrightarrow n^i < n^j$ $(i, j \in \mathcal{N})$. $\diamond$

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[48] An example is $U = x^\alpha y^{1-\alpha} + \phi(s)$ with $0 < \alpha < 1$ and $\phi : Q \to \mathcal{R}$ a strictly increasing function.

Proof. First statement – by contradiction. Suppose $n^i < n^j$ and $w^i \geq w^j$. Because $L^j \leq L^i$, we have $n^i, n^j \in X^i \cap X^j$. Writing $s_i := s^i(\mathbf{n})$ and $s_j := s^j(\mathbf{n}) > s_i$, we also have $n^i \in V^i_{s_i} \cup V^j_{s_i}$ and $n^j \in V^i_{s_j} \cup V^j_{s_j}$. Therefore, $n^i \in W^{ij}_{s_i}$ and $n^j \in W^{ij}_{s_j}$. By Lemma 51 in Appendix B, it holds $u^j(n^j, s_j) - u^j(n^i, s_i) > 0$. According to condition 39 with i and j interchanged, this implies $u^i(n^j, s_j) - u^i(n^i, s_i) > 0$. But this contradicts Lemma 51.

Second statement. For "$\Leftarrow$", see above. As for "$\Rightarrow$", suppose $w^i < w^j$. By Proposition 40, i and j are heterogeneous players. Because $\mathbf{n}$ is a fully diverse equilibrium, it follows that $n^i \neq n^j$. So $n^i < n^j$ or $n^i > n^j$. But $n^i > n^j$ is impossible because of the first statement. ■

In this setting, we can examine the impact on savings behaviour. We begin with fully diverse equilibria, which is the type of configuration studied by Frank (1985b) and also by Hopkins and Kornienko (2004a). It is convenient to use some old notation: for all $i \in \mathcal{H}_k$ $(k = 1, \ldots, M)$, we write $\hat{x}_k := \hat{x}^i$ and $\bar{x}_k := \bar{x}^i$. So condition 38(ii) implies that the optimal consumption at a given status level s depends on income according to $\hat{x}_1(s) < \cdots < \hat{x}_M(s)$. Now suppose the income differences are so large that for all $k = 1, \ldots, M-1$,

$$\bar{x}_k(s, 0) < \hat{x}_{k+1}(s)$$

$(s \in Q \backslash \{0\})$.[49] Then all Nash equilibria are fully diverse equilibria (Proposition 30). Even a continuum of fully diverse equilibria exists where the members of income class $\mathcal{H}_k$ have consumption c_k and status s_k given by

$$c_k \in [\hat{x}_k(s_k), \bar{x}_k(s_k, s_{k-1} + \frac{1}{N-1})] \text{ and } s_k = q_{\#\mathcal{H}_1 + \ldots + \#\mathcal{H}_k}$$

$(k = 1, \ldots, M)$. We know that equilibria with lower consumption levels, and thus higher saving rates, are Pareto superior and that the equilibrium with $c_k = \hat{x}_k(s_k)$ is Pareto efficient (Corollary 35).

Following Frank (1985b), let us compare this outcome with the case where consumers can*not* alter their status, being now determined by their rank in the income distribution. This case might reflect the situation where personal incomes are publicly observable. The idea is then that if incomes (and savings) are not public knowledge, present consumption as an observable normal good may signal personal income and thus shape the social hierarchy. With rank determined by income, the status of consumer i of income class $\mathcal{H}_k$ just equals

$$s^i = F_{\mathbf{w}^i}(w^i) = q_{\#\mathcal{H}_1 + \ldots + \#\mathcal{H}_k}.$$

His present consumption follows as $\hat{x}_k(q_{\#\mathcal{H}_1 + \ldots + \#\mathcal{H}_k})$.

The Pareto-efficient equilibrium above clearly coincides with our benchmark and we can say that the other equilibria exhibit overconsumption or undersaving. Contrary to Frank 's findings, therefore, status seeking need not always be inefficient or cause lower saving rates. It seems not right to qualify the Pareto-efficient equilibrium as simply being a borderline case. If we accept that the ordinal status game especially applies to small local groups and consider that, among a limited number of people, the probability that two of them have the same income is relatively small, the Pareto-efficient equilibrium may actually be the only equilibrium that exists (Theorem 25).

Also the claim that, under status seeking, there is a negative relation between saving rate and income does not hold here. In the Pareto-efficient equilibrium, the relation is the same as

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[49] Of course, $\hat{x}_k(s) \leq \bar{x}_k(s, 0)$ (Lemma 4). If $U(\cdot, \cdot; s)$ is homogeneous of degree one, $\hat{x}(s, w)$ is linear in wages, so the inequalities in the text can always be met for large enough income differences.

in the reference situation. The indeterminacy of the other equilibria precludes, in principle, any assessment of the relation between saving rate and income.

A special feature of the inefficient fully diverse equilibria is that the social ranking of individuals is the same as in the benchmark. Although everyone consumes more and saves less to improve his social status, in the end his relative position is stuck to his rank in the income distribution. Formally, we have for each equilibrium $\mathbf{n} \in E^{(M)}$

$$s^i(\mathbf{n}) = q_{\#\mathcal{H}_1 + \ldots + \#\mathcal{H}_k} \quad (i \in \mathcal{H}_k;\ k = 1, \ldots, M).$$

It suggests the picture of a positional treadmill where everyone runs faster but stays in the same place.

This picture alters when the differences between income classes are smaller, and the Nash equilibria take the form of clustering equilibria. Now there is at least one income class whose members really "catch up with the Joneses" of the next income class – the imitation of role models is complete. In formal terms, an income class $\mathcal{H}_k$ exists such that

$$s^i(\mathbf{n}) > q_{\#\mathcal{H}_1 + \ldots + \#\mathcal{H}_k} \quad (i \in \mathcal{H}_k).$$

It means that the social hierarchy becomes a squeezed version of the income hierarchy of our benchmark. Just as fully diverse equilibria, clustering equilibria are generally not unique, and equilibria with lower consumption levels imply a Pareto improvement (Propositions 27 and 36). The difference with fully diverse equilibria comes forward when we look at the behaviour of a specific cluster of income classes.

So suppose the cluster partition of a Nash equilibrium contains a cluster $\mathcal{C}_k = \mathcal{H}_m \cup \mathcal{H}_{m+1} \cup \cdots \cup \mathcal{H}_n$, where $k \leq m < n \leq M$. From Proposition 27, the consumption c_k and social status s_k of its members are such that

$$c_k \geq \hat{x}_n(s_k) \text{ and } s_k = q_{\#\mathcal{H}_1 + \ldots + \#\mathcal{H}_n}.$$

If $c_k > \hat{x}_n(s_k)$, the clustering equilibrium is not Pareto efficient. Even if $c_k = \hat{x}_n(s_k)$, we know that there may be room for Pareto improvement, because the lower income classes of the cluster may be better off by choosing a quantity of consumption somewhere between c_{k-1} and c_k.[50] Therefore, a clustering equilibrium is generally not Pareto efficient (Proposition 36).

Compared with our reference situation, any clustering equilibrium shows overconsumption and undersaving. Indeed, for each $i \in \mathcal{H}_p \subseteq \mathcal{C}_k \backslash \{\mathcal{H}_n\}$ we have

$$c_k \geq \hat{x}_n(s_k) = \hat{x}_n(q_{\Sigma_{j=1}^{n} \#\mathcal{H}_j}) > \hat{x}_p(q_{\Sigma_{j=1}^{n} \#\mathcal{H}_j}) \geq \hat{x}_p(q_{\Sigma_{j=1}^{p} \#\mathcal{H}_j}).$$

Only the members of the highest income class $\mathcal{H}_n$ may choose their optimal amount (viz. if $c_k = \hat{x}_n(s_k)$). Moreover, because the overconsumption of lower income classes is greater, saving rates fall with income. To see this, suppose two income classes $\mathcal{H}_p, \mathcal{H}_r \subseteq \mathcal{C}_k$ with $p < r$, then

$$c_k - \hat{x}_p(q_{\Sigma_{j=1}^{p} \#\mathcal{H}_j}) > c_k - \hat{x}_r(q_{\Sigma_{j=1}^{r} \#\mathcal{H}_j})$$

which, after some manipulations, implies

$$1 - \frac{p_x \hat{x}(q_{\Sigma_{j=1}^{p} \#\mathcal{H}_j}, w_p)}{w_p} - (1 - \frac{p_x c_k}{w_p}) > 1 - \frac{p_x \hat{x}(q_{\Sigma_{j=1}^{r} \#\mathcal{H}_j}, w_r)}{w_r} - (1 - \frac{p_x c_k}{w_r}).$$

Each side of the inequality shows the positive difference between the saving rates of the benchmark and the clustering equilibrium for an income class, and the gap is wider for the lower

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[50] See the discussion after Proposition 36 and also Example 37.

income class $\mathcal{H}_p$. For example, if the saving rates of the benchmark do not depend on income, the inequality becomes

$$1 - \frac{p_x c_k}{w_p} < 1 - \frac{p_x c_k}{w_r},$$

thus showing a lower saving rate for income class $\mathcal{H}_p$.

All in all, Frank's assertions about how status seeking affects saving are really relevant for clustering equilibria. Such equilibria are Pareto inefficient, except in borderline cases, and exhibit lower saving rates that, within clusters of income classes, indeed fall with income. On a more basic level, it appears that more than fully diverse equilibria the configuration of clustering equilibria captures the sociological notion of status seeking.

6 Concluding remarks

Above we analyzed a non-cooperative game in strategic form, where each player's payoff depends on his action and his social status, which is given by the fraction of players who take a lower or equal action than him. We focused on the relation between the degree of heterogeneity among status-seeking players and the distribution of their Nash equilibrium actions. Although some hard technicalities could not be avoided, we hope to have shown that the game-theoretic approach to status seeking is preferable to the usual partial-equilibrium approach, which is inadequate to deal with the existence and nature of Nash equilibria. Our approach also stands out because it did not use the heuristic assumption of a continuum of players. The latter is common in the literature on status, even though it is difficult to reconcile with the widely accepted view that people's sensitivity to status is particularly relevant in small local environments. The finding of multiple Nash equilibria and different types of Nash equilibria brought forward that concerns for relative position can integrate both economic and sociological explanations of human behaviour. Especially clustering equilibria capture this point, and we discussed an example where the number of social classes (people with the same status) is smaller than the underlying number of income classes. Another key result of the analysis was that, in contrast to what is usually claimed, status seeking need not always be socially inefficient.

One question remained unanswered: does each ordinal status game have a Nash equilibrium? The answer is yes in case of only two players, but the specific discontinuities of the payoff functions did not allow us to use the standard existence theorems to prove the general case. (However, we did derive two technical results that establish necessary and sufficient conditions for the existence of a Nash equilibrium (given a cluster partition and a set of level data).)

Future research might also try to generalize the game we studied here. One issue is whether a two-goods exchange economy, based on our two-goods game, would alter the main conclusions. The two-goods exchange economy has proved to be an instructive tool for obtaining fundamental insights about taxation and tax policy, so this setting also lends itself well to the evaluation of the policy proposals by Frank (1997, 2005) and others to levy taxes on positional goods or to introduce a progressive consumption tax.

Two other issues follow from the belief that the game particularly applies to a reference group of limited size. Suppose that players decide on more than one positional good, where each positional good is related to a specific reference group (reference groups may overlap). A player's personal valuation of his social status then depends on the relative positions he occupies in different reference groups (e.g., sports club, neighborhood, circle of friends). An interesting question is whether there are plausible shapes of the payoff functions that allow us to generalize our qualitative results in a straightforward fashion.

Regarding the other possibility, note that our analysis implicitly assumed that all individuals belong to one and the same reference group, and particularly that there is no way out. In reality, people are often free to choose their own associates and thus able to choose their reference groups. Although tied to his family, a person can usually select his neighbors by the house he buys and his co-workers by the job he applies for. Clearly, endogenizing the formation of reference groups, the central theme of Frank (1985a), would make the analysis complete. Yet extending the ordinal status game by letting each player choose his reference group, without introducing additional structure, is problematic. Since homogeneous players can enjoy maximum status by forming their own reference group, the typical outcome is that there are as many reference groups as there are homogeneity classes. This is not very realistic (see also Frank, 1984; Postlewaite, 1998). Switching from one reference group to another often takes time and money; sometimes it is simply impossible. Also, and this possibility suggests options for future research, there may be benefits related to a heterogeneous reference group.[51] For example, flocking together increases the size of the reference group, which may entail the fruits from external economies of scale. Joining a higher income class may give access to local public goods supplied to this class. Or, finally, if homogeneity classes differ in relative productivity, the gains from cooperation may compensate the losses in status for those specializing in low status tasks.

<*The appendices are availiable on request*>

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[51] Frank (1984) argues that heterogeneity can persist if high-status groups compensate low-status groups for not dropping out of the reference group. In our setting with assumption (4), such a compensation scheme will not come about, because the best option for each homogeneity class is to separate.

References

[1] Akerlof, G.A. (1997), Social distance and social decisions, *Econometrica* 65, 1005-1024.

[2] Bagwell, L.S., and B.D. Bernheim (1996), Veblen effects in a theory of conspicuous consumption, *American Economic Review* 86, 349-373.

[3] Ball, S., C. Eckel, P.J. Grossman and W. Zame (2001), Status in markets, *Quarterly Jourmal of Economics* 115, 161-188.

[4] Boskin, M. and E. Sheshinski (1978), Optimal redistributive taxation when individual welfare depends upon relative income, *Quarterly Journal of Economics* 92, 589-601.

[5] Clark, A.E. and A.J. Oswald (1998), Comparison-concave utility and following behaviour in social and economic settings, *Journal of Public Economics* 70, 133-155.

[6] Congleton, R.D. (1989), Efficient status seeking: externalities, and the evolution of status games, *Journal of Economic Behavior and Organization* 11, 175-190.

[7] Corchón, L. (2001), *Lecture notes in Economics and Mathematical Systems*, Springer, Berlin.

[8] Corneo, G. and O.Jeanne (1998), Social organization, status, and savings behavior, *Journal of Public Economics* 70, 37-51.

[9] Corneo, G. and O.Jeanne (2001), Status, the distribution of wealth, and growth, *Scandinavian Journal of Economics* 103, 283-293.

[10] Duesenberry, J.S. (1949), *Income, Saving and the Theory of Consumer Behaviour*, Harvard University Press, Cambridge.

[11] Dynan, K.E., J. Skinner and S.P. Zeldes (2004), Do the rich save more? *Journal of Political Economy* 112, 397-444.

[12] Easterlin, R.A. (2001), Income and happiness: towards a unified theory, *Economic Journal* 111, 465-484.

[13] Frank, R.H. (1984), Are workers paid their marginal product? *American Economic Review* 74, 549-571.

[14] Frank, R.H. (1985a), *Choosing the Right Pond: Human Behaviour and the Quest for Status*, Oxford University Press, London and New York.

[15] Frank, R.H. (1985b), The demand for unobservable and other nonpositional goods, *American Economic Review* 75, 101-116.

[16] Frank, R.H. (1997), The frame of reference as a public good, *Economic Journal* 107, 1832-1847.

[17] Frank, R.H. (2005), Progressive consumption taxation as a remedy for the U.S. savings shortfall, *The Economists' Voice 2*, Iss. 3(2).

[18] Hirsch, F. (1976), *Social Limits to Growth*, Harvard University Press, Cambridge.

[19] Hopkins E. and T. Kornienko (2004a), Running to keep in the same place: Consumer choice as a game of status, *American Economic Review* 94, 1085-1107.

[20] Hopkins E. and T. Kornienko (2004b), Consumption, status, and redistribution, *working paper.*

[21] Ireland, N., 1994, On limiting the market for status signals, *Journal of Public Economics* 53, 91–110.

[22] Jones, S.R.G. (1984), *The Economics of Conformism*, Blackwell, Oxford.

[23] Konrad, K.A. (1990), Status präferenzen: soziobiologische Ursachen, Statuswettrüsten und seine Besteuerung, *Kyklos* 43, 249-272.

[24] Kosicky, G. (1987), A test of the relative income hypothesis, *Southern Economic Journal* 54, 422-434.

[25] Layard, R. (1980), Human satisfactions and public policy, *Economic Journal* 90, 737-750.

[26] Layard, R. (2005), *Happiness. Lessons from a New Science.* The Penguin Press. London.

[27] Leibenstein, H. (1950), Bandwagon, snob and Veblen effects in the theory of consumer's demand, *Quartely Journal of Economics* 64, 183-207.

[28] Loch, C.H., B.A. Hubermann and S. Stout (2000), Status competition and performance in work groups, *Journal of Economic Behavior and Organization* 43, 35-55.

[29] Luttmer, E.F.P. (2005), Neighbors as negaives: relative earnings and well-being, *Quartely Journal of Economics* 119, 963-1002.

[30] Mason, R. (1998), *The Economics of Conspicuous Consumption: Theory and Thought since* 1700, Elgar. Cheltenham.

[31] Morgenstern, O. (1948), Demand theory reconsidered, *Quartely Journal of Economics* 62, 165–201.

[32] Neumark, D. and A. Postlewaite (1998), Relative income concerns and the rise in married women's employment, *Journal of Public Economics* 70, 157-183.

[33] Oxoby, R.J. (2003), Attitudes and allocations: status, cognitive dissonance, and the manipulation of attitudes, *Journal of Economic Behavior and Organization* 52, 365-385.

[34] Oxoby, R.J. (2004), Cognitive dissonance, status and growth of the underclass, *Economic Journal* 114, 727-749.

[35] Postlewaite, A. (1998), The social basis of interdependent preferences, *European Economic Review* 42, 779-800.

[36] Ridgeway, C. and H.A. Walker, 1995, Status structures, in: Cook, K., G. Fine and J. House (Eds.), *Sociological Perspectives on Social Psychology*, Allynn & Bacon, Newton MA.

[37] Robson, A.J. (1992), Status, the distribution of wealth, private and social attitudes to risk, *Econometrica* 60, 837-858.

[38] Schor, J.B. (1998), *The Overspent American*, Basic Books, New York.

[39] Solnick, S.J. and D. Hemenway (1998), Is more always better?: A survey on positional concerns, *Journal of Economic Behavior and Organization* 37, 373-383.

[40] Varian, H. (1974), Equity, envy and efficiency, *Journal of Economic Theory* 9, 63-91.

[41] Veblen, T. (1899), *The Theory of the Leisure Class*, Macmillan, New York.